NOTES ON COMPACT MODULI OF K3 SURFACES

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LECTURE 1: MODULI OF K3 SURFACES

0.1. **Beginnings of moduli.** The study of a moduli space of Riemann surfaces of genus g was initiated by Riemann, who was the first to perform the heuristic calculation that the space of such surfaces depends on $3g-3$ complex parameters, or "moduli."

In the century following, the beautiful theory of the moduli space $\mathcal{M}_g$ was uncovered by work of Klein, Poincaré, Teichmüller, and others. Formalizing intuitions about $\mathcal{M}_g$ played an integral role in many fields of mathematics, for instance in defining topological spaces, manifolds, and groups. Understanding Riemann surfaces was a major motivation of Klein's *Erlangen program*, which sought to understand geometry in terms of the group of symmetries of those geometries.

The study of K3 surfaces, while relatively more recent than that of curves, also spurred many important developments in mathematics. Their study was initiated by the Italian school of algebraic geometry, in the early 20th century. It was already understood in the Erlangen program that algebraic curves split into three broad categories: The positively curved case $g=0$, the flat case $g=1$, and the negatively curved case $g \geq 2$.

Enriques, Castelnuovo, and later Kodaira, extended these results to surfaces, categorizing them by their *Kodaira dimension*

$$\kappa(X) := \dim \bigoplus_{m\geq 0} H^0(X, mK_X) - 1.$$

The $\kappa = -\infty$ surfaces are ruled, the $\kappa = 0$ surfaces are Calabi-Yau, the $\kappa = 1$ surfaces are elliptically fibered, and the remaining "general type" surfaces have $\kappa = 2$. Within the $\kappa = 0$ surfaces are those covered by an abelian surface (the abelian and bielliptic surfaces), and those covered by a K3 surface (the Enriques and K3 surfaces).

Definition 0.1. A *K3 surface* X is a compact complex surface, which is simply connected and has trivial canonical bundle $K_X = \mathcal{O}_X$.

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Example 0.2. Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree 4, then the adjunction formula implies that $K_X = (K_{\mathbb{P}^3} + 4H)\big|_X = \mathcal{O}_X$. By the Lefschetz hyperplane theorem, $\pi_1(X) = \pi_1(\mathbb{P}^3) = 0$ and hence X is a K3 surface. Other examples include complete intersections $X_{2,3} \subset \mathbb{P}^4$, $X_{2,2,2} \subset \mathbb{P}^5$, and double covers $X \to \mathbb{P}^2$ branched over a sextic curve.

Some of the earliest K3 surfaces to be considered were the "Flächen vierten Grades mit sechzehn singulären Punkten" of Kummer in 1884. These are the quotients of principally polarized abelian surfaces by negation.

Theorem 0.3 (Enriques, 1909)**.** *For all $g \geq 2$, there are surfaces $X \subset \mathbb{P}^g$ of degree $2g - 2$ embedded by a complete linear system, with trivial canonical bundle $K_X = \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$.*

Theorem 0.4 (Severi, 1909)**.** *For each $2d = 2g - 2$, the number of moduli of such surfaces is* 19.

Example 0.5. Counting parameters for quartic hypersurfaces, we have the space of quartic polynomials on $\mathbb{P}^3$, which has dimension $\binom{7}{4} = 35$, minus the space of linear transformations $\mathrm{GL}_4(\mathbb{C})$, which has dimension 16. Thus, the parameter count is $35 - 16 = 19$.

Some major results came from the work of Kodaira and Kuranishi, who developed the theory of deformations of complex structures.

Theorem 0.6 (Kodaira, 1964)**.** *All K3 surfaces are deformation equivalent, and the space of complex deformations of a K3 surface is* 20-*dimensional.*

In particular, all K3 surfaces are diffeomorphic, and more weakly, have the same cohomology ring. We have $H^i(X, \mathbb{Z}) = 0$ for $i = 1, 3$, $H^i(X, \mathbb{Z}) = \mathbb{Z}$ for $i = 0, 4$. Most important is that $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$ and that $H^2(X, \mathbb{Z})$ has a perfect, symmetric bilinear intersection form, isometric to the unique even unimodular lattice $II_{3,19}$ of signature $(3, 19)$.

The second cohomology $H^2(X, \mathbb{Z})$ admits a weight 2 *polarized Hodge structure*: An orthogonal decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

for which $H^{p,q} = \overline{H^{q,p}}$ and $(H^{2,0})^\perp = H^{2,0} \oplus H^{1,1}$. We additionally have $x \cdot \bar{x} > 0$ for any nonzero $x \in H^{2,0} = H^0(X, \Omega^2) \simeq \mathbb{C}$.

Theorem 0.7 (Siu, 1983)**.** *All K3 surfaces are Kähler.*

The term *K3 surface* was coined by Weil in 1958, who named them after the three mathematicians: Kähler, Kummer, Kodaira and after the K2 mountain (because of its beauty).

0.2. **Torelli theorems.** An important development in the theory of moduli of curves was the result of Torelli that a curve could be reconstructed from essentially linear-algebraic data.

Theorem 0.8 (Torelli, 1913). *Let* $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ *be a standard system of curves on a Riemann surface* C *and let* $(\omega_1, \ldots, \omega_g)$ *be the basis of the abelian differentials on* C *for which* $\int_{\alpha_i} \omega_j = \delta_{ij}$*. Then the isomorphism type of* C *is uniquely recoverable from the symmetric* $g \times g$ *period matrix* $(\int_{\beta_i} \omega_j)$*.*

In modern terminology, we would say: C can be recovered from the polarized Hodge structure on $H^1(C, \mathbb{Z})$.

The central result to the understanding of moduli of K3 surfaces is provided by an analogous "Torelli theorem" of Piatetski-Shapiro and Shafarevich:

Theorem 0.9 (Piatetski-Shapiro, Shafarevich 1973). *Let* $\{\alpha_1, \ldots, \alpha_{22}\}$ *be a standard system of generators of* $H_2(X, \mathbb{Z})$*. Let* Ω *be the non-vanishing holomorphic* 2*-form on* X *for which* $\int_{\alpha_1} \Omega = 1$ *(the standard system may be chosen so that such that this integral is nonzero and* $\alpha_1 \cdot \alpha_2 = 1$*). Then the isomorphism type of* X *is uniquely recoverable from the vector*

$$(\int_{\alpha_3} \Omega, \ldots, \int_{\alpha_{22}} \Omega) \in \mathbb{C}^{20}.$$

Another way to phrase this theorem is as follows: Two K3 surfaces X and X' are isomorphic if and only if there exists an isometry

$$\phi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

for which $\phi(H^{2,0}(X)) = H^{2,0}(X')$. We call such an isometry a *Hodge isometry*. In addition to this statement concerning the isomorphism type of a single surface, we also have a *local Torelli theorem*: This essentially says that the complex deformation space of X is locally isomorphic to the period space $\mathbb{C}^{20}$.

Definition 0.10. The *period domain* of K3 surface is

$$\mathbb{D} := \mathbb{P}\{x \in II_{3,19} \otimes \mathbb{C} \,\big|\, x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

A *marking* of a K3 surface is an isometry $\phi\colon H^2(X, \mathbb{Z}) \to II_{3,19}$ and the *period* of a marked K3 surface (X, ϕ) is $\phi(H^{2,0}(X)) \in \mathbb{D}$.

Theorem 0.11 (Local Torelli Theorem). *Let* (X, ϕ) *be a marked K3 surface and let* $\mathfrak{X} \to U \subset H^1(X, T_X) \simeq \mathbb{C}^{20}$ *be the universal deformation, which exists by the results of Kuranishi and Kodaira, due to the fact that* $h^0(X, T_X) = h^2(X, T_X) = 0$*. Then all fibers of* $\mathfrak{X} \to U$ *have a marking, induced by the marking of* $X = \mathfrak{X}_0$*. The resulting period map* $U \to \mathbb{D}$ *is a local isomorphism.*

Let $L \to X$ be a primitive ample line bundle. Then $c_1(L)\cdot[\Omega] = 0$ and furthermore, $c_1(L) \in H^2(X, \mathbb{Z})$. We can refine the notion of a marking for a pair (X, L) by choosing $v \in II_{3,19}$ primitive with $v \cdot v = 2d$ and require a marking of (X, L) to send $\phi\colon c_1(L) \mapsto v$. Then the period mapping lands rather in

$$\mathbb{D}_{2d} = \mathbb{P}\{x \in v^\perp \otimes \mathbb{C} \,|\, x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

Let $\Gamma_{2d} := \{\gamma \in O(II_{3,19}) \,|\, \gamma(v) = v\}$. The great upshot of restricting to polarized K3 surfaces is that now Γ_{2d} acts properly discontinuously on $\mathbb{D}_{2d}$. Hence, given any family $(\mathfrak{X}, \mathfrak{L}) \to S$ over a base S, we have a canonical defined *period map*

$$S \to \mathbb{D}_{2d}/\Gamma_{2d}.$$

Theorem 0.12. *The moduli space of polarized smooth K3 surfaces of degree* $2d$ *is a Zariski open subset of* $\mathbb{D}_{2d}/\Gamma_{2d}$*. If we allow the polarized K3 surface to have ADE singularities, then the coarse moduli space of degree* $2d$ *K3 surfaces is exactly* $\mathbb{D}_{2d}/\Gamma_{2d}$*.*

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