Forschungsseminar: Applications of the étale fundamental group to algebraic geometry

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Let X/k be a smooth variety over a field and suppose $x \in X(k)$ is a rational point. A choice of an algebraic closure $k \hookrightarrow k^{\text{sep}} \hookrightarrow \bar{k}$ therefore determines a geometric point $\bar{x} \to X$. There is an outer action of $\text{Gal}(k^{\text{sep}}/k)$ on the étale fundamental group $\pi_1(X_{\bar{k}}, \bar{x})$. (From now on, we suppress the basepoint of all fundamental groups.) Very loosely, the goal of the seminar is to understand structural features of this action, especially when k is a finitely generated over a prime field.

More specifically, we wish to understand features and properties of those local systems which come from geometry. For our purposes, we think of local systems as continuous representations of the étale fundamental group. (More will be said on this in the early lectures.) Here is an ad hoc definition.

Definition 0.1. *Let X/k be a smooth variety over a field k and fix a geometric point $\bar{x}$ to define the étale fundamental group $\pi_1(X, \bar{x})$. Let $E/\mathbb{Q}_\ell$ be a finite extension, where ℓ is prime. An E-*local system on X *is a continuous homomorphism $\pi_1(X, \bar{x}) \to GL_d(E)$, where the right hand side has the ℓ-adic topology, for some $d \geq 1$. A $\overline{\mathbb{Q}}_\ell$-*local system on X *is an E-local system on X for some finite extension $\overline{\mathbb{Q}}_\ell \supset E \supset \mathbb{Q}_\ell$.*

For instance, if $k = \mathbb{C}$ and $X = \mathbb{G}_m$, we have that $\pi_1(X) \cong \hat{\mathbb{Z}}$. An abstract character $\chi \colon \hat{\mathbb{Z}} \to \mathbb{Q}_\ell^\times$ is continuous if and only if (any) generator of $\hat{\mathbb{Z}}$ is mapped to an element $\mathbb{Z}_\ell^\times$, i.e., if the entire image is contained in the set of ℓ-adic numbers of absolute value 1.

Definition 0.2. *Let X/k be a smooth variety over a field. Let L be a $\bar{\mathbb{Q}}_\ell$ local system, where ℓ is prime to the characteristic of p. We say that L is* of geometric origin *(or geometric, or motivic) if threre exists an open dense $U \subset X$, a smooth projective morphism $f \colon \mathcal{Y} \to U$, and an integer $i \geq 0$ such that $L|_U$ is a subquotient of:*

$$R^i f_*(\bar{\mathbb{Q}}_\ell)$$

.

When $k = \mathbb{C}$, then sub-quotient may be replaced by "summand" by a theorem of Deligne.

The orienting questions are the following:

- Can we "characterize" those local systems which arise from algebraic geometry? What properties do they have?
- How many local systems come from geometry? How are they located in moduli (i.e., in character varieties or their rigid counterparts).
- When there are "enough" motivic local systems, what general facts can we deduce about local systems?

There are many interesting (and wide open) conjectures about these questions, but there have been a number of recent developments in the last few years.

Here are several of the results we will prove in this course. The first result is a basic structural property of motivic local systems: the eigenvalues of the parallel transport operator associated to loops around a boundary divisor at infinity all are roots of unity. (Note that the theorem is vacuous if X is proper.)

Theorem 0.3. *(Grothendieck quasi-unipotent monodromy) Let X/k be a smooth variety and let L be a motivic local system on X. Then L has* quasi-unipotent monodromy at ∞.

To understand the next result, we first recall a celebrated result of Mazur: any $\mathbb{Q}$-rational torsion point of an elliptic curve $E/\mathbb{Q}$ has order dividing 12. The geometric torsion conjecture is the following: the torsion part of the Mordell-Weil group of a family of abelian varieties over a complex quasiprojective curve is uniformly bounded in the genus (or even gonality) of the curve.[1]

We state the next result in terms of complex algebraic geometry and the topological fundamental group; strikingly, the proof goes through arithmetic.

Theorem 0.4. *([L21, Corollary 1.1.15] with antecedent [L18, Corollary 1.6]) Let $X/\mathbb{C}$ be a smooth connected quasiprojective variety. Then there exists a number $N = N(X) > 0$ such that if:*

$$\rho\colon \pi_1^{top}(X^{an}) \to GL_n(\overline{\mathbb{Z}})$$

is a local system that arises from geometry and is trivial mod M for some integer $M > N$, then ρ is trivial.

Note that N is independent of n, the rank of the representation. In particular, the above result has the following corollary. Fix a curve $X/\mathbb{C}$. Then there is an integer N such that any non-isotrivial family of abelian varieties $A \to X$ that is generically simple cannot have a full collection of M-torsion sections for any $M > N$.[2]

The following theorem is a strengthening of a corollary to a theorem of Deligne. Deligne's argument uses Hodge theory; Litt's argument uses as inputs some properties of p-adic dynamical systems and also Lafforgue's solution of the Langlands correspondence for function fields.[3]

Theorem 0.5. *([L21, Corollary 1.1.5]) Let $X/\mathbb{C}$ be a smooth connected variety and let $n > 0$ be an integer. Then the collection of representations*

$$\rho\colon \pi_1^{top}(X^{an}) \to GL_n(\mathbb{Q}_\ell)$$

of geometric origin up to isomorphism is finite.

Litt in particular proves that the collection of $\overline{\mathbb{Q}}_\ell$ local systems of geometric origin have no limit points in the ℓ-adic topology. On the other hand, there has been recent speculation that the collection of local systems of geometric origin may be dense in the Euclidean topology.

Time permitting, we could also discuss some subset of the papers [EK20a, EK20b, P20].

Outline

1. (21.04) Introduction (Raju).

   Introduce classical fundamental group. Define local systems as representations of fundamental groups, with emphasis on the story over $\mathbb{C}$. Explain how the cohomology of a family of algebraic varieties gives rise to a local system as above. State the goal theorems of the seminar.

   Define the étale fundamental group. State Grothendieck's short exact sequence and the corresponding *outer* action of $\mathrm{Gal}(k)$ on $\pi_1^{\mathrm{geo}}(X)$. Give several examples as to how much information the *arithmetic* fundamental group controls, e.g. Mochizuki's theorems.

2. (28.04) **Goal**: Finish background on étale fundamental group from last time. Explain the pro-ℓ and $\mathbb{Q}_\ell$-pro-unipotent group rings of a profinite group.

   Say a word about the equivalence between "ℓ-adic local systems" and representations of π_1. Given a family $f\colon Y \to X$ and a prime ℓ not equal to the characteristic of k, state that $R^i f_* \mathbb{Q}_l$ is a $\mathbb{Q}_\ell$ local system on X, and hence is equivalent to a continuous, finite dimensional representation of $\pi_1(X)$.

[1] In symbols, the conjecture says the following. Given $g, d > 0$, there exists an integer $N = N(g, d)$ such that for any quasi-projective curve $C/\mathbb{C}$ whose compactification $\bar{C}$ has genus g and any family of abelian varieties $A_C \to C$ with no "fixed part", the group of torsion sections has size no greater than N.

[2] As a fun exercise, prove this in the particular case of families of elliptic curves!

[3] For the experts: Lafforgue comes in only to show that absolutely irreducible arithmetic local systems with finite order determinant over a curve over a finite field are pure of weight 0.

Give [L18, Definition 2.1]: define the pro-ℓ group ring of a profinite group Γ, the augmentation ideal, and the unipotent completion.[4] Explain that modules over the pro-ℓ group ring are the same as continuous ℓ-adic representations of Γ and that the modules over the unipotent completion are the same as unipotent representations of the group.

Explain [L18, Example 2.3] and [L18, Proposition 2.4]. (Remark that when X/k is smooth projective, the abelianization and ℓ-ification of this action is (dual to) the Galois representation $H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$.)

3. (05.05) **Goal**: Quasi-unipotent monodromy theorem.

   Present Grothendieck's proof over a mixed characteristic local field.[5] In particular, explain the ramification filtration: $P \subset I \subset \pi_1$, see [dJcourse, Definitions 8.1, 8.2]. I recommend reading [dJcourse, Lectures 14, 15]. The proofs of the main theorem are in [dJcourse, 15.3, 15.4]. Another possible reference: [dJ01, Lemma 2.12].

   Explain why this doesn't *immediately* imply the result over $\mathbb{C}$ (too many roots of unity!) [dJcourse, p. 43-44]. Time permitting, state Abhyankar's lemma and Néron desingularization, which together do recover the result over $\mathbb{C}$. [dJcourse, Lecture 16]. (This is too optimistic, don't worry about this.)

4. (12.05) **Goal**: [L18, Theorem 2.8]. Define *arithmetic representations*. State Theorem 1.4.[6]

   Define the weight filtration, as in [L18, Section 2.2]. State verbally the relevant content of [BL21, Remark 3.10] as to what this weight filtration morally measures.

   State [L18, Lemma 2.9] (standard), sketch the proof of [L18, Lemma 2.10]. State Semisimplicity theorem [L18, 2.12] (no proof) and then combine to sketch [L18, Theorem 2.8], a.k.a. "existence of Bogomolov elements". Time permitting, try to give some explanation for why Bogomolov elements exist.

5. (19.05) **Goal** [L18, Theorem 3.6]. More coloquially, define the "convergent group rings". (Pure group theory)

   Explain [L18, example 3.1], give [L18, Definition 3.2]. Give [L18, Example 3.3]. Prove [L18, Proposition 3.4]. State [L18, Theorem 3.6] and sketch as much of the proof as you can. Make sure to mention the intuitive content that the denominators of the eigenvectors don't grow too quickly.

6. (02.06) **Goal:** Main theorems of [L18].

   Put the results of last two lectures together: prove [L18, Theorem 1.2]. (Note that [L21, 1.1.13], which is a strengthening of this result, just takes as input a result of Serre. The proofs are the same; prove whichever one you want.) Explain [L18, Examples 4.4-4.6], state [L18, Question 4.7].

7. (09.06) **Goal:** Deformation of representations. (Pure group theory, source is [K]. An alternative source is Mazur's article in the volume on Fermat's last theorem [CSS97, Ch. VIII].)

   Define deformation problem. State and prove [K, 1.2.1]. State and prove [K, 1.3.1]. State [K, 1.4.1]. If you can, explain the relationship with the unipotent completion.

8. (16.06) **Goal:** Statement of de Jong's conjecture. Relation to deformation ring being finite flat.

   State [dJ01, Conjecture 1.1]. State [dJ01, Theorem 1.2 (i)] or its more precise cousin [dJ01, Theorem 3.5]. Go through [dJ01, Section 3] and prove as much as you can of the theorem. (I will take 10-15 minutes at the end of this lecture to explain the relation to a conjectured density of motivic local systems.)[7]

9. (23.06) **Goal:** Pseudo-representations. (Pure group theory.)

   Main source: [K, Lecture 2]. Emphasize that the deformation problem for pseudo-representations works more generally when the residual representation is not absolutely irreducible. Also the relevant facts about moduli on [L21, p. 8-9] (just state).

[4] An excellent source for the unipotent completion of a group is [BF, Section 3.3].
[5] The first written reference I am aware for this is [ST68, Appendix], which is extremely readable.
[6] Note that the same proof in the paper gives [L21, 1.1.14, 1.1.15] using a letter of Serre; if time permits, state these.
[7] Note that de Jong's conjecture is now a theorem. We will not cover the proof of this in the seminar.

10. (30.06) **Goal:** State [L21, Theorem 1.1.3]. Explain how this generalizes Deligne's finiteness theorem: if $X/\mathbb{C}$ is a smooth quasi-projective variety, then there are only finitely many $\mathbb{Q}$-local systems on X that come from algebraic geometry. State and prove [L21, Lemma 3.2.1].

11. (07.07) **Goal:** Prove [L21, 1.1.3, 1.1.5]. State [L21, Corollary 4.1.6, Remark 4.1.7], and give some small indication for what result in p-adic dynamical systems are necessary for this (i.e., what [L21, Lemma 4.1.1] buys you.) Sketch how we can use these results to derive [L21, 1.1.3].

12. (14.07) **Goal:** State and prove [L21, Theorem 1.1.11].

    (Optional, we can also try to pivot to one of [EK20a, EK20b, P20] instead.)

References

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