Introduction to geometric local systems 2

Last time, we briefly discussed de Jong's conjecture, resolved by Gaitsgory. Combined w/ the work of L. Lafforgue, we have the following conclusion.

Thm Let $U/\overline{\mathbb{F}_q}$ be a hyperbolic curve.

Let $\mathcal{S} := \left\{ \begin{array}{l} L \ \overline{\mathbb{Q}}_\ell\text{-local systems} \\ \text{on } U/\overline{\mathbb{F}_q} \text{ w/ fixed} \\ \text{rank, bounded ramification @} \\ \infty, \text{ fixed local monodromies} \end{array} \right\}$

$\mathcal{S}^{geo} = \{ L \text{ as above, of geo. origin} \}$

$\mathcal{S}^{geo} \subset \mathcal{S}$

Then ① $\mathcal{S}^{geo}$ is infinite

② ~ " $\mathcal{S}^{geo}$ is Zariski dense in $\mathcal{S}$ "

Rmk: This theorem is crazy!!
For instance, we have the exmpl.

Example Let $\lambda \in \overline{\mathbb{F}}_p$. and set.

$$\mathcal{U} := \mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}.$$

Then $\exists$ ∞'ly many (isogeny classes of) simple abelian schemes

$$\begin{array}{c} A_{\mathcal{U}} \\ \downarrow \\ \mathcal{U} \end{array}$$

of GL_2-type and semi-stable reduction.

skip?

+ Moreover, there is a sense in which this ∞, appropriately weighted, is independent of λ

Q: Direct proof / construction?

They are all of Hilbert modular type.

Rmk Example true for any affine hyperbolic curve.

Q: Why do we care about the theorem?!

A: ☆ Using dJ's conj., Drinfeld proved semi-simplicity + Hard Lefschetz thms for perverse sheaves in char 0.

☆ Using dJ + companions f... dJ-Esnault proved that

$\pi: X \to Y$ (quasi proj. normal)

L_Y is s.s. $\mathbb{C}$-local system

Then $\pi^{*} L_Y =: L_X$ is semi-simple on X.

Q: Is CordJG true for other base fields? E.g.

Question: Let $X/\mathbb{C}$ be smooth. Let $\mathrm{Char}^B(X)$ be a character variety (moduli of $\pi_1(X) \to GL_N$)

Is the set of pts of geo origin dense?!?

L-L prove that such a statement is false in general for low rank.

Notation C sm. proj. curve/$\mathbb{C}$ genus g

$x_1, \dots, x_N$ distinct pts of C.

$U := C \setminus \{x_1, \dots, x_N\}$

Thm LL1 Let $(C, x_1, \dots, x_N)$ be analytically very general in $\mathcal{M}_{g,n}$. Let L be a local system of geo origin on $\mathcal{U}$ w/ ∞-monodromy. Then

$$\operatorname{rank} L \geq 2\sqrt{g+1}$$

<u>Slogan</u>: "a very general curve admits <u>NO</u> low rank local systems of geo origin"

In fact, they prove that if $\mathbb{L}$ is an $\mathcal{O}_K$-local system on U,

s.t. $\forall \iota : K \hookrightarrow \mathbb{C}$,

$\iota_{\mathbb{L}}$ underlies a PVHS,

then:
$$\operatorname{rank}(\mathbb{L}) \leq 2\sqrt{g+1}$$
$$\Rightarrow \mathbb{L} \text{ has finite monodromy}$$

Note: such a statement requires the # field K: every curve (w/ even Euler char) has a "natural" rank 2 $\mathbb{R}$-local system underlying an $\mathbb{R}$ VHS:

$$\rho: \pi_1(C) \longrightarrow SL_2(\mathbb{R})$$

$$\begin{array}{c} \mathfrak{h} \circlearrowleft PSL_2(\mathbb{R}) \\ \downarrow \\ C \end{array}$$

NOTE: Any statement like Thm LL1 requires a rank bound by Kodaira-Parshin trick.

<u>Motivation</u> $\mathcal{M}_h$ not proper.

Do there exist complete curves in $\mathcal{M}_h$?

<u>Answer</u> $h \geq 3$ YES:

$$\mathcal{M}_h \hookrightarrow \underbrace{\mathcal{A}_h \hookrightarrow \mathcal{A}_h^{*}}_{\text{boundary has codim } h}, \qquad \text{BB}$$

take general intersection of ample divisors.

Explicit construction?

$$X \longrightarrow \mathcal{M}_h$$

$$\iff \exists \text{ curve } \begin{array}{c} \mathcal{Y} \\ \downarrow \\ X \end{array} \text{ of genus } h.$$

For every pt $p \in X$, want a curve $\mathcal{Y}_p$.

How on earth are we going to construct such a thing?

IDEA: MAKE IT RELATED TO X!!

$$\begin{tikzcd} & Y_x \arrow[dl] \arrow[dr] & \\ * & & X \end{tikzcd}$$

S_3 cover, triple branched over x.

Q: Does this give me my family $\begin{matrix} Y \\ \downarrow \\ X \end{matrix}$?

A: NO! triple cover not uniquely determined! $\exists$ $3^{2\,\mathrm{genus}(X)}$ choices

$\rightsquigarrow$

$$\underbrace{\mathcal{X} \longrightarrow \mathcal{C}}_{} = \mathcal{M}_{g,1}$$

$$\downarrow$$

$$\mathcal{M}_g$$

relative curve,
fibers not connected!
$\sim 3^{2g}$ components

rel genus $\sim 6g$

$\rightsquigarrow$ $\exists$ geometric local system over general curve of rank $O(g \cdot 3^{2g})$

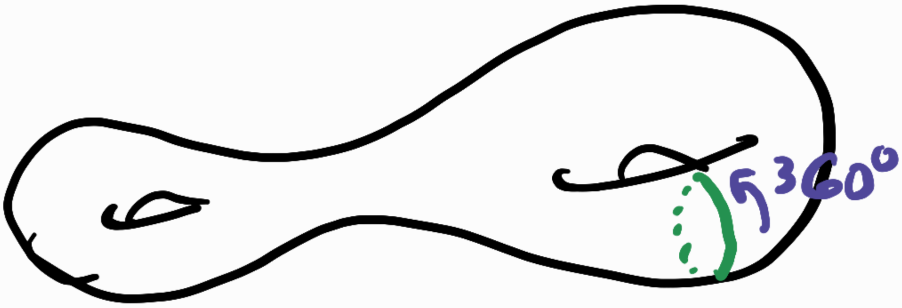
To state next result, need to recall the mapping class group.

$\Sigma_g :=$ compact orientable top. surface of genus g.

$$\Sigma_{g,n} := \Sigma_g \setminus \{x_1,\dots,x_n\}$$

$$MCG_{g,n} := \underbrace{\mathrm{Homeo}^+(\Sigma_g,\{x_1,\dots,x_n\})}_{\text{oriented homeomorphisms } \Sigma_g \to \Sigma_g \text{ preserving } x_1,\dots,x_n} \Big/ \text{isotopy}$$

$$= \pi_0\left(\mathrm{Homeo}^+(\Sigma_g, \{x_1,\dots,x_n\})\right)$$



"Dehn twist"

Skip

$$\boxed{1 \to \text{Torelli} \longrightarrow MCG_g \twoheadrightarrow Sp_{2g}(\mathbb{Z}) \to 1}$$

<u>Thm</u> (Dehn)

MCG_g generated by finite set of Dehn twists around simple closed curves.

<u>Obs</u>: $MCG_{g,n}$ does not "act" on $\pi_1(\Sigma_{g,n}, x)$

However, it "acts by outer automorphisms":

$$MCG_{g,n} \longrightarrow Out(\pi_1(\Sigma_{g,n}, x))$$

Hence $MCG_{g,n}$ does <u>NOT</u> act on $Hom(\pi_1(\Sigma_{g,n}, x), GL_n(\mathbb{C}))$

But, it does act on

$$\mathrm{Char}(\Sigma_{g,n}) := \mathrm{Hom}\big(\pi_1(\Sigma_{g,n}, x), GL_N(\mathbb{C})\big) \Big/ \text{conj.}$$

(in general,

$$\begin{array}{ccc} \mathrm{Aut}(G) & \curvearrowright & \mathrm{Hom}(G, H) \\ \downarrow & & \downarrow \\ \mathrm{Out}(G) & \curvearrowright & \mathrm{Hom}(G,H) \Big/ \text{conj by } H \end{array}$$

let $\tilde{\varphi} \in \mathrm{Aut}(G)$ $\quad \overset{g^{-1} - g}{\mathrm{Ad}_g \circ \tilde{\varphi}}$

$\tilde{\varphi} . \rho \, (x) := \rho(\tilde{\varphi}^{-1}(x))$

$\mathrm{Ad}_g \circ \tilde{\varphi} . \rho(x) := \rho(\; g \; \tilde{\varphi}^{-1}(x) g^{-1})$

<u>Def</u> A rep

$$\rho : \pi_1(\Sigma_{g,n}) \to GL_N(\mathbb{C})$$

is <u>MCG-finite</u> (or <u>canonical</u>)

if $MCG_{g,n}$ orbit of ρ is finite.

<u>Idea</u>: MCG finite representations morally correspond to (log) flat connections on the universal curve

$$\begin{array}{c} \mathcal{U} := \mathcal{C} \setminus \{x_1, \dots, x_N\} \\ \downarrow \\ \mathcal{M}_{g,n} \end{array}$$

"alg isomonodromic deformation"

Reason: Let $\mathbb{T}_g$ be Teichmüller space

$$\begin{tikzcd} \mathbb{T}_g \arrow[d] \\ M_g \end{tikzcd}$$

universal cover

$\pi_1(M_g)$ is MCG_g

$\pi_1(M_{g,n})$ almost $MCG_{g,n}$

$$\begin{tikzcd} \mathcal{D}^{univ} \arrow[r] \arrow[d] & \mathbb{T}_g \arrow[d] \\ \mathcal{C}^{univ} \arrow[r] & M_g \end{tikzcd}$$

universal

Given a top $\mathbb{C}$-local system L on Σ_g, L canonically extends isomonodromically to a local system $\mathcal{L}$ on $\mathcal{D}^{univ}$

($\rightsquigarrow$ relative flat connection.)

We say L admits a (versal) alg isomonodromic deformation if $\mathcal{L}$ on $\mathcal{D}^{univ}$ descends to $\mathcal{C}^{univ}$ (or, more precisely, to a scheme $\mathcal{C}' \underbrace{\longrightarrow}_{\text{étale \& dominant}} \mathcal{C}^{univ}$)

where

—

<u>Idea</u>: the orbit of $[\rho]$ under $MCG_{g,n}$ is "monodromy of monodromy"

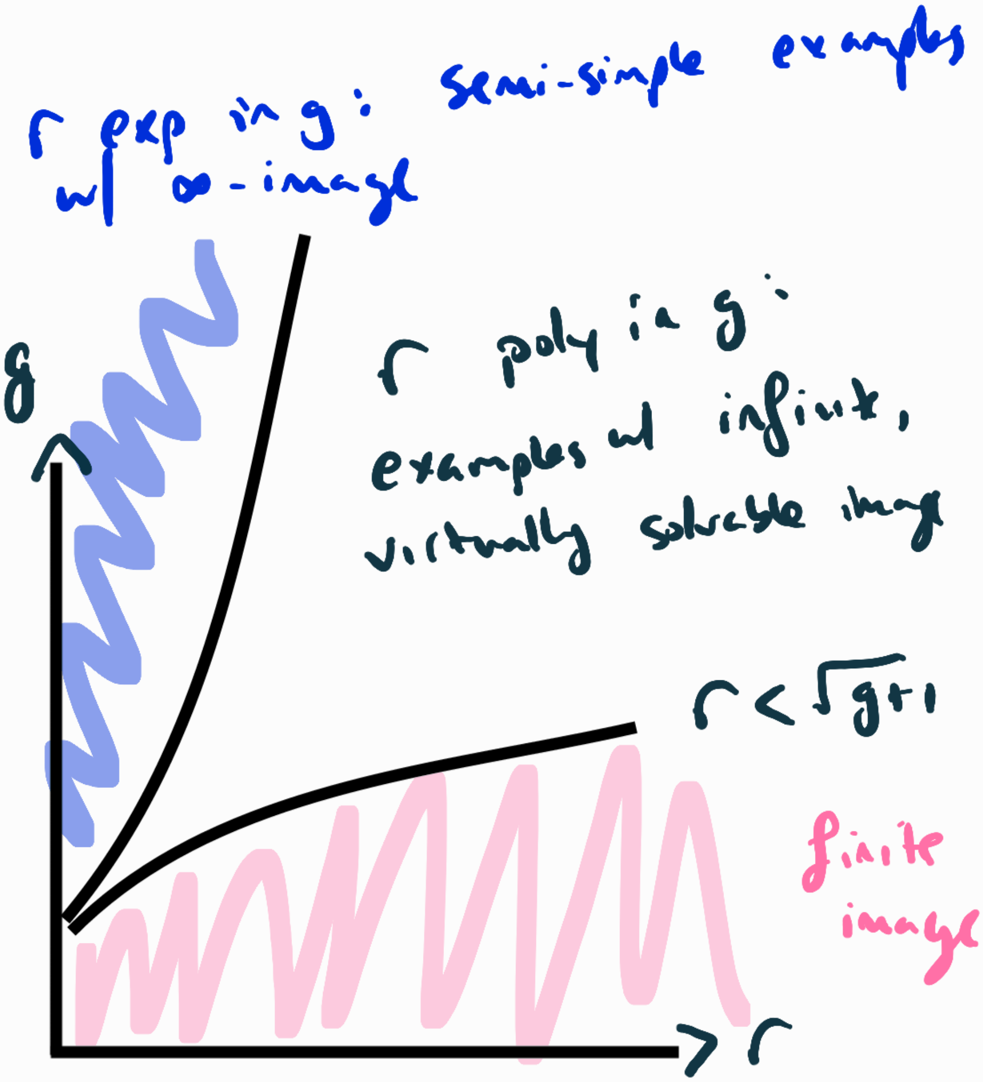
<u>Philosophy</u>: this should generally be big.

Thm LL2 $\rho: \pi_1(\Sigma_{g,n}) \to GL_N(\mathbb{C})$ be MCG-finite. If ρ has ∞-monodromy, then

$$N > \sqrt{g+1}$$

Slogan: canonical representations have large rank.

GEOGRAPHY OF MCG-finite reps:



Cor of Thm 11.2

$\pi_1(\Sigma_{g,1}) = F_{2g}$

Let $\rho : F_{2g} \longrightarrow GL_r(\mathbb{C})$

be a rep such that

$\mathrm{Out}(F_{2g}).[\rho]$ is a finite set

(i.e., $[\rho]$ has finite orbit under

$\mathrm{Out}(F_{2g})$.)

Then if r is small (i.e., $r < \sqrt{g+1}$), then ρ is

finite.

Note: can describe generators of $\mathrm{Out}(F_n)$, see Remark 1.6.3 of [LL22c].

Thm LL3: $(C, x_1, \dots, x_n) / K$ w.f.g.

be "general", i.e.,

$$\operatorname{Spec} K \to \mathcal{M}_{g,n} \ni \eta$$

is dominant

Let $\rho : \pi_1(U_{\bar{K}}) \to GL_n(\overline{\mathbb{Q}}_\ell)$.

Suppose

① ρ is of geo origin

② ρ has ∞-image

Then $\operatorname{rank}(L) \geq \sqrt{g+1}$

OPTIONAL

Putnam–Wieland

Let H be a finite gp, $\Sigma_{g',n'} \to \Sigma_{g,n}$ an unbranched H cover.

$$MCG_{g,n+1} \curvearrowright \pi_1(\Sigma_{g,n}, x) \xrightarrow{\phi} H$$

Let $\Gamma \subseteq MCG_{g,n+1}$ be stabalizer of $\ker(\phi)$.

$\rightsquigarrow \Gamma \curvearrowright H_1(\Sigma_{g',n'}, \mathbb{C})$

$\rightsquigarrow \Gamma \curvearrowright H_1(\Sigma_{g'}, \mathbb{C})$

"fill in punctures"

For any $\rho: H \to GL_N(\mathbb{C})$,

$H_1(\Sigma_{g',n'}, \mathbb{C})^{\rho}$ is isotypic component (H acts on $\Sigma_{g',n'}$)

<u>Def</u>

Fix $g \geq 2$, $n \geq 0$, and H as above.

We say $PW^{H}_{g,n}$ holds if:

$\forall$ H-cover $\Sigma_{g',n'} \to \Sigma_{g,n}$

$$\Gamma \curvearrowright H_1(\Sigma_{g',n'}, \mathbb{C})$$

has no vectors of finite orbit.

Conj (PW) $\forall g \geq 2, n \geq 0$, H

$PW^{H}_{g,n}$ holds.

Cor of Thm 4.2

For fixed g, n, $PW^{H}_{g,n}$ holds

for any group H w/ following

property:

- all irreps ρ of H
have rank $< g$.

Sketch of Pf of ThmCC1

- $((C, x_1, \dots, x_N), (E, \nabla)$ on C w/
  - rank $E < 2\sqrt{g+1}$
  - reg. sing, nilp. res.

Suppose isomonodromic def to any general nearby n-pted curve underlies $\mathbb{C}$PVHS

Then we want to prove (E, ∇) has unitary monodromy (i.e. $KS \equiv 0$)

- For any such as above, we will show that

☆ If $(\mathcal{E}, \nabla)$ is the isomonodromic def, then $\mathcal{E}$ is semi-stable on the (analytically) very general curve.

If ☆ holds, then:

- Hodge filtration consists of one piece.
- $\Rightarrow \exists$ definite Hermitian form (polarization) preserved by monodromy
- $\Rightarrow$ monodromy is unitary

Therefore, reduced to proving ☆.

This follows from

Theorem 1.3.4 (LL22a), which morally says the following:

Let (E, ∇) be a $^{(\text{log})}$ flat connection on an analytically very general curve $(C, x_1, \ldots, x_n)$. Then E is semi-stable.

Pf: Def thy of $(E, \nabla, \mathrm{Fil}_{HN})$ + Clifford theory for vector bundles

Key: "non-GGG" Lemma, (5.1.3, 5.1.4 of [LL22a])

To prove Thm 1.1.2

- cohomology vanishing for unitary local systems on versal families of curves ($R^1\pi_*$(unitary) underlies PVHS)

- MCG-finite $\Rightarrow$ strongly coh. rigid on $\begin{array}{c} \mathscr{C}^\circ \\ \downarrow \\ \Delta \end{array}$

  $\underset{\text{E-G, K-P}}{\Longrightarrow}$ integral

(these will imply the result for unitary local systems)

- In general, deform to PVHS. over a fiber, will have unitary monodromy. Use above techniques to show that MCG-finite reps don't have non-MCG-finite deformations.

Talks

① Intro

② Atiyah bundles + Isomonodromic deformation

③ "non-GGG lemma"

very important!

(4) Prove Thm 1.3.4 of LL22a: analysis of Fil_{HN} & isomonodromic deformation.

(4) CPVHS, Higgs, positivity.

- Explain proof of main thm of [LL22a]

(5) Basics on $MCG_{g,n}$, canonical reps. Explain equivalence w/ local systems on versal families.

(6) Use non-GGG to prove cohomology rank bound for a unitary local

system over a versal family.

7. (coho) rigidity (using non-GGG). Deduce main thm for unitary MCG-finite reps.

8. thm for semi-simple reps, arithmetic application

9. Putnam-Wieland