Landesman - Litt 5   03.01.23

. **Def** $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$

A $\mathbb{C}$-local system $\mathbb{L}$ on $U := C - \{x_1, \ldots, x_n\}$ is of geometric origin if $\exists\ f : Y \longrightarrow U$ smooth projective / $\mathbb{L}$ is a subquotient of $R^i f_* \mathbb{C}$.

$\Updownarrow$ direct summand

. **Thm 1.2.4**: $\mathcal{T}_{g,n} \xrightarrow{\pi} \mathcal{M}_{g,n}$, with (g,n) hyperbolic $[\,g \geq 2$, or $g=1, n>0$, or $g=0, n>2\,]$

$\exists$ countable union W of strict complex analytic subsets of $\mathcal{T}_{g,n}$ such that for $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} - \pi(W)$

any $\mathbb{C}$-local system of geometric origin with ∞ monodromy has rank $\geq 2\sqrt{g+1}$

"analytically very general"

CVHS: X complex connected manifold.

. A $\mathbb{C}$VHS on X is $(V = \oplus V^{p,q}, D)$ $\mathcal{C}^\infty$-$\mathbb{C}$ vector bundle, D a flat connection satisfying Griffiths transversality: $D(V^{p,q}) \subset A^{1,0}(V^{p,q}) \oplus A^{0,1}(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1}) \oplus A^{0,1}(V^{p+1,q-1})$

$\Big(\Leftrightarrow \ \big(\underset{\ker D \otimes \mathcal{O}_X}{\overset{\shortparallel}{E}} = (V, \bar\partial)$ holom. $\supset (F^p = \bigoplus_{r \geq p} V^{r,s}, \bar\partial)$, $\nabla F^p \subset \Omega^1 \otimes F^{p-1}$

and similarly $\bar F^q := \bigoplus_{s \geq q} V^{r,s}$ antiholomorphic, $\nabla \bar F^q \subset \bar F^{q-1} \otimes \overline{\Omega^1_S} \Big)$

- A polarization on $(V = \oplus V^{p,q}, D)$ is $\psi : V \otimes \overline{V} \to \mathbb{C}$ hermitian D-flat making $\overset{\perp_\psi}{\oplus} V^{p,q}$ and such that $(-1)^p \psi > 0$ on $V^{p,q}$

[Ex: $f : Y \to X$ smooth projective; $\underset{\mathbb{Z}}{\mathbb{V}} = R^i f_* \mathbb{Z}$ polarizable $\mathbb{Z}$VHS $\Rightarrow$ $V_{\mathbb{C}}$ polarizable $\mathbb{C}$VHS; any subquotient is a direct summand and is a polarizable $\mathbb{C}$VHS.]

Suppose now $X \hookrightarrow \overline{X} \supset Z$ smooth algebraic log pair.
$\uparrow$ SNC divisor

Def (Deligne canonical extension)

(E, ∇) flat holom. vector bundle on X with unipotent monod. at ∞.

Its Deligne canonical extension $(\overline{E}, \overline{\nabla} : \overline{E} \to \overline{E} \otimes \Omega^1_{\overline{X}}(\log Z))$ is the unique v.b. $\overline{E}$ on $\overline{X}$ with logarithmic connection along Z equipped with an iso $(\overline{E}, \overline{\nabla})_{|X} \xrightarrow{\sim} (E, \nabla)$, characterized by the property that its residues along Z are nilpotent.

Prop: $X \hookrightarrow \overline{X} \supset Z$ smooth alg. log pair.

H ample line bundle on $\overline{X}$

$(E, F^\bullet, \nabla)$ polarizable $\mathbb{C}$VHS, $(\overline{E}, \overline{\nabla})$ Deligne extension.

Then:

1/ $\mathbb{V} := \operatorname{Ker} \nabla$ is semi-simple ← Schmid $/\mathbb{R}$ + Deligne 87

2/ $\mathbb{V} \simeq \bigoplus_i \mathbb{L}_i \otimes W_i$, where the $\mathbb{L}_i$'s are pairwise non isomorphic irreducible $\mathbb{C}$-loc. syst. on X, W_i vector space. Each $\mathbb{L}_i$ underlies a polar. $\mathbb{C}$VHS, each W_i carries a polarized $\mathbb{C}$HS, unique up to shifting and grading, and comp. with $\mathbb{V}$. (Deligne 87)

3/ $c_{\mathbb{Q}}(\overline{E}) = 0$ (total chern class) (Esnault - Viehweg)

4/ $F^\bullet$ extend to $(\overline{E}, \overline{F}^\bullet, \overline{\nabla})$ satisfying Griffiths transv. (Schmid nilpotent orbit theorem)

5/ The Higgs bundle $(\bigoplus_i gr^i_{F^\bullet} \overline{E}, \theta)$ is H-polystable of degree 0 (Simpson 90 for curves)

$$\left[\ \overline{\nabla} F^p \subset F^{p-1} \otimes \Omega^1(\log Z) \quad \Rightarrow \theta := gr\, \overline{\nabla} : gr^p_F \longrightarrow gr^{p-1}_F \otimes \Omega^1(\log Z)\ \ \mathcal{O}_{\overline{X}}\text{-linear}\right]$$

Prop: (Cor. 1.3.9) $(\overline{E}, \overline{F}^\bullet, \overline{\nabla})$ as above.

$\overline{E}$ semistable $\Leftrightarrow$ the $\mathbb{C}$VHS is unitary.

Proof. This is due to Griffiths (curvature formula).

Let i max. / $F^i\overline{E}$ $(= gr^i_F \overline{E})$ non-trivial.

Claim: if $\theta_i \neq 0$ then $\deg F^i\overline{E} > 0$ (hence $\overline{E}$ not semistable)

[Indeed: $(\bigoplus gr^j_F \overline{E}, \theta) \twoheadrightarrow (\bigoplus_j gr^j_F \overline{E} / \bigoplus_{j\neq i} gr^j_F \overline{E}, \overline{\theta}) = (gr^i_F \overline{E}, 0)$ so $\deg gr^i_F \overline{E} \geq 0$ as $(\bigoplus gr^j_F, \theta)$ semi-stable

If $\deg gr^i_F \overline{E} = 0$ then $(gr^i_F \overline{E}, 0)$ direct summand by polystability. Ruled out by $\theta_i \neq 0$.]

So $\theta_i = 0$ and $F^i\overline{E}$ is unitary direct summand. Iterate.

Thm 1.2.8 $(C, x_1, \dots, x_n)$ hyperbolic of genus g.
(E,∇) flat with $\mathrm{rk}\, E < 2\sqrt{g+1}$ with regular sing. and unipotent mon. at ∞.
If an isom. def. of E to an analytically general curve underlies a $\mathbb{C}$VHS then (E,∇) is unitary

Proof Replacing $(C, x_1, \dots, x_n)$ with an analytic. general curve, (E,∇) is a $\mathbb{C}$VHS
By 1.3.6, E is semistable
So by prop. (E,∇) is unitary.

Proof of 1.2.5 For $K \subset \mathbb{C}$ # field and $\rho : \pi_1(U) \longrightarrow GL_n(\mathcal{O}_K)$ with ∞ image
$M_{g,n} \supset T_\rho := \{ [C', x_1, \dots, x_n] \mid \forall\, i : K \hookrightarrow \mathbb{C},$
the local syst. $\mathbb{V}_\rho \otimes_i \mathbb{C}$ on U' is a $\mathbb{C}$VHS $\}$

Claim: $T_\rho \subset$ strict closed analytic subset $T'_\rho \subset \mathcal{T}_{g,n}$
Otherwise: for an analytic v.g point $(C, x_1, \dots, x_n)$ $\mathbb{V}_{\rho,i}$ $\mathbb{C}$VHS.
By 1.2.8, $\mathbb{V}_{\rho,i}$ unitary
But then $\rho : \Gamma \longrightarrow GL_m(\mathcal{O}_K) \longrightarrow \prod_i GL_m(\mathbb{C})$ is unitary, so finite. Contradiction to ρ has ∞ image.

Take $W = \bigcup_{\rho \text{ as above}} T'_\rho$ $\square$