Atiyah bdle and isomonodromic deformation

30.11.2022

First GOAL of Seminar:

Thm (Landesman – Litt) local system of rank $< 2\sqrt{g+1}$ on "analytically very general" curve of genus g is not of geometric origin.

Goal today. introduce enough to prove toy version.

§1. Atiyah bdle • $C/\mathbb{C}$ curve, E (algebraic) vector bundle on C.

• T_C = tangent sheaf.

Defn (i) $\mathrm{Diff}^1(E) = \left\{ \begin{array}{l} C^\infty,\ \mathbb{C}\text{-lin. endomorphisms } \tau : E \to E \text{ s.t.} \\ \forall \text{ (local) section } f \text{ of } \mathcal{O}_C,\ v \mapsto \tau(fv) - f\tau(v) \\ \quad \text{is } \mathcal{O}_C\text{-linear endomorphism of } E \end{array} \right\}$

$= \text{first order differential operators } E \to E.$

(ii) $At(E) \subset \mathrm{Diff}^1(E)$ is the sub-sheaf s.t. $v \mapsto \tau(fv) - f\tau(v)$ is given by mult^n by $\underbrace{\delta_\tau(f)}$, a local section of $\mathcal{O}_C$

(iii) Suppose $P^\bullet$ = filtration on E. Then $At(E, P^\bullet) \subset At(E)$ is the sheaf of τ's preserving $P^\bullet$.

(straightforward to check)

$$\boxed{\begin{array}{l} \textbf{Prop. } \exists \text{ SES} \\ \quad 0 \to End_{P^\bullet}(E) \to At(E, P^\bullet) \xrightarrow{\delta} T_C \to 0, \quad \text{where } \delta(\tau)(f) = \delta_\tau(f). \\ (*) \text{ Moreover, splittings } q^\nabla \text{ of this} \leftrightarrow \text{flat connections } \nabla \text{ on } E. \quad \left[\begin{tikzcd} At(E) \arrow[r, bend left, "\delta"] & T_C \arrow[l, bend left, "q^\nabla"] \end{tikzcd} \right] \end{array}}$$

Rmk For $D \subset C$ a reduced divisor, have $T_C(-D) \subset T_C$ and Atiyah bdle (def. as pullback)

$$0 \to End_{P^\bullet}(E) \to At_{(C,D)}(E, P^\bullet) \xrightarrow{\delta} T_C(-D) \to 0$$

And splittings $\leftrightarrow$ connection w/ RS along D.

Rmk ⚠ • alternatively let $\begin{array}{c}\Pi \\ \downarrow p \\ C\end{array}$ = frame bundle of E.

$= \underline{\mathrm{Hom}}(\mathcal{O}_C^{\oplus n}, E)$.

= principal GL_n-bdle

then have SES on Π

$$0 \to T_{\Pi/C} \longrightarrow T_\Pi \to p^* T_C \to 0$$

which is GL_n-eq, and descends to SES on C.

The descended SES is precisely

$$0 \to \mathrm{End}(E) \to \mathrm{At}(E) \to T_C \to 0$$

from above.

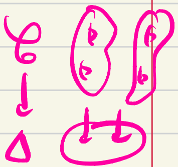
• Similarly, for filtration $P^\bullet \subset E$, use $\Pi_P = \{\text{frames compatible with } P^\bullet\}$, principal P-bundle where $P \subset GL_n$ is parabolic preserving $P^\bullet$.

2. Isomonodromy

Setup $\mathcal{C}, \Delta$ cx mfds with

$\pi : \mathcal{C} \to \Delta$ proper submersion, fibers connected of dim one, $s_i : \Delta \to \mathcal{C}$ ($i = 1, \dots, n$) disj. sections.

w/ Δ contractible, with $C = \pi^{-1}(0)$, $\mathcal{D} = \bigcup \operatorname{im}(s_i)$, $D = C \cap \mathcal{D}$.



Lemma (E, ∇) flat vector bdle on C with reg sing along D extends uniquely to $(\mathcal{E}, \tilde{\nabla})$ on $\mathcal{C}$.

(**Pf sketch** $\pi_1(C \setminus D) \xrightarrow{\sim} \pi_1(\mathcal{C} \setminus \mathcal{D})$ so (E, ∇) extend to $\mathcal{C} \setminus \mathcal{D}$. Then use Deligne's canonical extn. to get $\mathcal{E}$.)

Defn (2.2.3) Above is the isomonodromic deformation. For $\Delta = T_{S,n}$, we call it the univ. iso def.

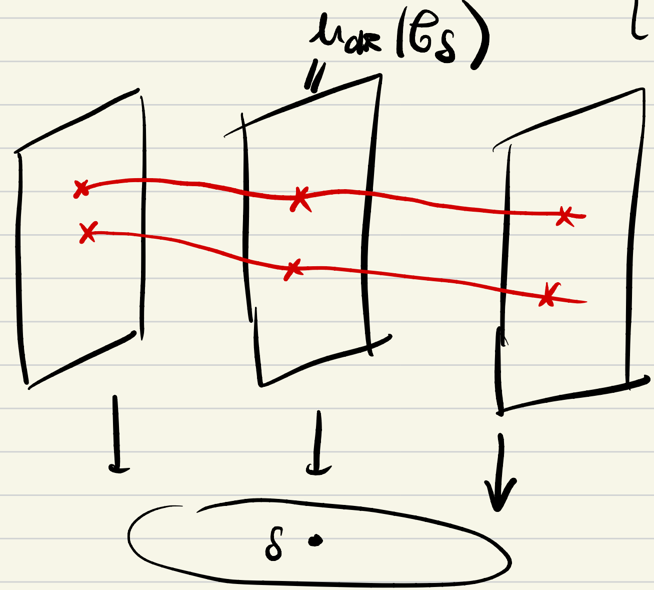
Example. Family of families $\mathcal{X} \xrightarrow{\pi} \mathcal{C} \to \Delta$, with π proper sm. and for each $\delta \in \Delta$ get flat bdle $H^i_{dR}(\mathcal{X}_\delta / \mathcal{C}_\delta)$ on $\mathcal{C}_\delta$.

Geometric interpretation of isomonodromic def. (Bruno)

[$\mathcal{C}_\delta$ = fiber of $\mathcal{C}$ over $\delta \in \Delta$]

consider moduli of flat bdles

$$\begin{array}{c}\mathcal{M}_{dR}(\mathcal{C}/\Delta) \\ \downarrow \\ \Delta.\end{array}$$



= leaves of foliation.

- fibers locally canonically isomorphic $\xrightarrow{\sim}$ $\mathcal{M}_B(\mathcal{C}_\delta)$ = character variety
  so gives foliation on $\mathcal{M}_{dR}(\mathcal{C}/\Delta)$ (= "non-abelian connection" on the space $\mathcal{M}_{dR}(\mathcal{C}/\Delta)$)
- leaves of foliation = iso def.

§3. Deformation theory.

Let $(C, D, E, p^\bullet)$ be as above.

Consider the deformation problems:

Defn. For (A, m) Artin $\mathbb{C}$-alg,

- $\mathrm{Def}_{(C,D)}(A) = \left\{ \begin{array}{l} \text{flat morphism } (\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Spec}\, A \\ f : C \hookrightarrow \mathcal{C} \\ \text{inducing iso'n } C \xrightarrow{\sim} \mathcal{C} \times \mathrm{Spec}\, A/m \\ \text{taking } D \text{ isomorphically to } \mathcal{D} \times \mathrm{Spec} \end{array} \right\}$

- $\mathrm{Def}_{(C,D,E,p^\bullet)}(A) = \left\{ \begin{array}{l} (\mathcal{C}, \mathcal{D}),\ f \text{ as above} \\ + \text{ bdle } \mathcal{E} + \text{filtration} \ldots \end{array} \right\}$

Prop. (i) $\mathrm{Def}_{(C,D,E,P^\bullet)}(\mathbb{C}[\varepsilon]/\varepsilon^2) \xrightarrow{\sim} H^1(C, At_{(C,D)}(E,P^\bullet))$

(ii) the induced map
$$\mathrm{Def}_{(C,D)}(\mathbb{C}[\varepsilon]/\varepsilon^2) \cong H^1(T_C(-D)) \xrightarrow{g^\nabla} H^1(At_{(C,D)}(E))$$
is that given by iso.

(iii) if P deform under iso in the direction $\xi \in H^1(T_C(-D))$, then
$$\xi \in \ker\left(H^1(T_C(-D)) \longrightarrow H^1(At_{(C,D)}(E)/At_{(C,D)}(E,P))\right)$$

Rmks. ① Intuitively (forget $P^\bullet$)
$$0 \to End(E) \to At(E) \to T_C \to 0$$
and recall $H^1(End(E))$ = tangent space to def of E itself.

$H^1(T_C)$ = tgt space to def of C itself.

$\therefore$ $H^1(At(E))$ combines both deformations.

② Proof of (i): use frame bundle Π from above, get class $\in H^1(\Pi, T_\Pi)$ equivariant for Gl_n-action $\rightsquigarrow$ class $\in H^1(At(E))$ (and same for $At(E,P)$, etc).

Recall first goal of this series

Thm (Landesman–Litt) local system of rank $< 2\sqrt{g+1}$ on analytically very general curve is not of geometric origin.

Toy model **Claim** C curve with $g \geq 2$, V geometric rank 2 local system ($\leftrightarrow (E, \nabla)$) with ∞ monodromy, the univ iso def is not motivic.

Assume not, so generic def is motivic, so underlies VHS.

Pf. Have SES

$$0 \to T_C \xrightarrow{g^\nabla} At(E)/At(E,\nabla) \longrightarrow Q \to 0$$

$$= \underbrace{(Fil^1)^{\otimes -2}}_{\text{neg degree if not iso-trivial}} = L$$

Hodge theory

Note that Fil^1 deforms in all directions, so induced map $H^1(T_C) \to H^1(L)$ is identically zero by (iii) of Proposition above.

But have $0 \to H^0(Q) \to H^1(T_C) \to H^1(L) \to H^1(Q) = 0$, and hence

$$H^1(L) = 0.$$

$\therefore$ $H^0(L) = H^1(L) = 0 \Rightarrow \deg L = g-1$ (RR)

which contradicts $\deg L < 0$, as required.

$\square$

Note. the above actually shows that if V is of rank 2 and underlies VHS (and of ∞ monodromy), then the univ. isomonodromic deformation does not underlie a VHS.