Introduction to:

"Canonical Representations of surface groups"

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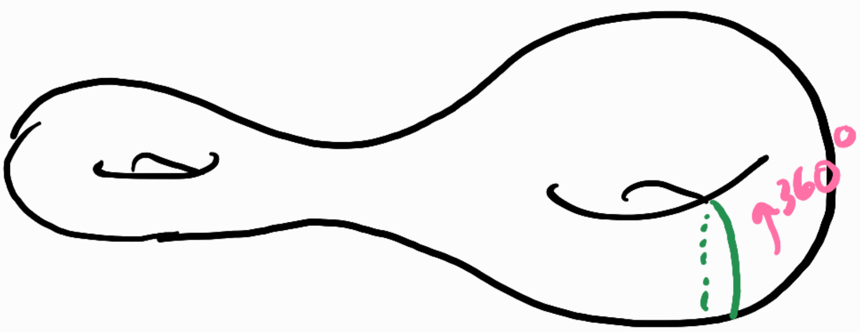
Let $\Sigma_g$ be compact, orientable, genus $g$ surface, $x_1, \dots, x_n \in \Sigma_g$

(Assume $(g,n)$ is hyperbolic)

$$MCG_{g,n} := \underbrace{\mathrm{Homeo}^+(\overbrace{\Sigma_g, \{x_1, \dots, x_n\}}^{\Sigma_{g,n}})}_{\text{orientation preserving homeomorphisms } \Sigma_g \to \Sigma_g \text{ sending } \{x_1,\dots,x_n\} \to \{x_1,\dots,x_n\}} / \text{isotopy}$$

$$= \pi_0(\mathrm{Homeo}^+(\Sigma_g, \{x_1, \dots, x_n\}))$$




"Dehn twist"

Thm (Dehn) $MCG_g$ generated by a finite collection of "Dehn twists" around simple closed curves.

$$\boxed{1 \to \text{Torelli} \to MCG_g \xrightarrow{\star} Sp_{2g}(\mathbb{Z}) \to 1}$$

Mysterious!
- $g=2$, infinite rank free group
- $g \geq 3$, finitely generated, unknown if finitely presentable.

$\star$ realizes the action of $Homeo^+$ (and hence $MCG_g$ on $H^1(\Sigma_g, \mathbb{Z})$

Def $1 \to \boxed{PMCG_{g,n}} \to MCG_{g,n} \to S_n \to 1$ is the "pure mapping class group", fixing the punctures.

Q: Does $MCG_g$ act on $\pi_1(\Sigma_g, x)$?

A: No, b/c it does not preserve $x$.

However, " the fundamental gp is well defined up to inner automorphisms "

$$\Longrightarrow \quad MCG_g \longrightarrow \mathrm{Out}(\pi_1(\Sigma_g, x))$$

(and similarly, $MCG_{g,n} \longrightarrow \mathrm{Out}(\pi_1(\Sigma_{g,n}, x))$)

Hence $MCG_{g,n}$ does <u>NOT</u> act on

$$\mathrm{Hom}(\pi_1(\Sigma_{g,n}, x), GL_r(\mathbb{C})),$$

but it does act on:

$$\mathrm{Char}^r(\Sigma_{g,n}) := \mathrm{Hom}(\pi_1(\Sigma_{g,n}, x), GL_r(\mathbb{C})) / \mathrm{conj}$$

[in general,

$$\mathrm{Aut}(G) \curvearrowright \mathrm{Hom}(G,H)$$

$$\downarrow$$

$$\mathrm{Out}(G) \curvearrowright \mathrm{Hom}(G,H)\big/ \text{conj. by } H$$

]

<u>Def</u> A representation $\rho: \pi_1(\Sigma_{g,n}, x) \to GL_r(\mathbb{C})$ is MCG-finite (or canonical) if the $MCG_{g,n}$ orbit of

$$[\rho] \in \mathrm{Char}^r(\Sigma_{g,n})$$

is finite

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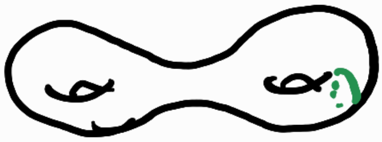
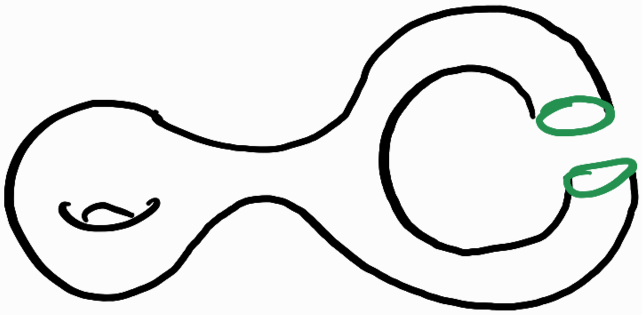
<u>Motivation</u>

$$\underbrace{T_g \longrightarrow M_g}_{MCG_g \simeq \pi_1(M_g)}$$

Intuition for

$$MCG_g \longrightarrow \pi_1(\mathcal{M}_g)$$





$\ni \theta \in [0, 2\pi]$

Varying $\theta$ gives us a map $S^1 \to \mathcal{M}_g$

(note that complex structure changes!)

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ \mathcal{M}_{g,n} \arrow[u, bend left, "s_1"'] \arrow[u, bend left=60, "s_n"] \end{tikzcd} \quad , \quad \mathcal{C}^0 = \mathcal{C} \setminus \cup(\mathrm{Im}(s_i)) = \mathcal{M}_{g,n+1}$$

$$\begin{tikzcd} \mathcal{C}^0 \arrow[dr] & \mathcal{M}_{g,n+1} \arrow[d] \\ & \mathcal{M}_{g,n} \end{tikzcd}$$

## Lemma 2.0.1

- $\pi_1(\mathcal{M}_{g,n}) \cong PMCG_{g,n}$

- The following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\Sigma_{g,n}) \arrow[r] & \pi_1(M_{g,n+1}) \arrow[r] & \pi_1(M_{g,n}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_{g,n}) \arrow[r] \arrow[u, equal] & PMCG_{g,n+1} \arrow[r] \arrow[u, "\simeq"] & PMCG_{g,n} \arrow[r] \arrow[u, "\simeq"] & 1
\end{tikzcd}$$

$$\underbrace{1 \to \pi_1(\Sigma_{g,n}) \to PMCG_{g,n+1} \to PMCG_{g,n}}_{\text{"Birman exact sequence"}},$$

- $PMCG_{g,n+1} \to PMCG_{g,n}$ induced from

$$\underbrace{\mathrm{Homeo}^+(\Sigma_{g}, x_1, \ldots, x_{n+1})}_{\text{preserves each}} \to \mathrm{Homeo}^+(\Sigma_{g}, \underbrace{x_1, \ldots, x_n}_{\text{puncture}})$$

- $\pi_1(\Sigma_{g,n})$ " = " $\pi_1(\Sigma_{g,n}, x_{n+1})$,

the map $\pi_1(\Sigma_{g,n}) \to PMCG_{g,n+1}$ is determined as:

$$[\underbrace{\gamma}_{\text{simple closed curve}}] \longmapsto \text{Dehn twist}(\gamma)$$

Question: Which representations are MCG-finite?!

Prop 2.1.2

Let $\Gamma \subseteq MCG_{g,n+1}$ be a finite index subgroup containing $\pi_1(\Sigma_{g,n})$.

Let $\rho : \Gamma \longrightarrow GL_r(\mathbb{C})$.

Then $\rho|_{\pi_1(\Sigma_{g,n})} : \pi_1(\Sigma_{g,n}) \longrightarrow GL_r(\mathbb{C})$

is MCG finite.

<u>Slogan</u>: restrictions of reps of $MCG_{g,n+1}$ to $\pi_1(\Sigma_{g,n})$ are MCG-finite.

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We wish to state a geometric incarnation.

<u>Def</u> Let 
$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ \mathcal{A} \arrow[u, bend left, "s_n"] \arrow[u, bend right, "s_1"'] \end{tikzcd}$$
be a genus $g$ curve w/ $n$-sections. We say the tuple $(\mathcal{C}, \mathcal{A}, s_1, \dots, s_n)$ is ==versal== if induced map
$$\mathcal{A} \longrightarrow M_{g,n}$$
is étale & dominant.

Notation $\mathcal{C}^\circ := \mathcal{C} \setminus \bigcup_{i=1}^{n} \operatorname{Im}(s_i)$

"the actual punctured curve over $\Delta$"

Prop 2.1.3

Let $\pi^\circ : \mathcal{C}^\circ \to \Delta$ be a versal family of $(g,n)$ curves. Let $C^\circ$ be any fiber of $\pi^\circ$. If

$$\rho : \pi_1(\mathcal{C}^\circ) \to GL_r(\mathbb{C})$$

is a rep, then $\rho|_{\pi_1(C^\circ)}$ is MCG-finite

Key fact • $\pi_1(\mathcal{S}) \longrightarrow \pi_1(\mathcal{M}_{g,n})$

has image a finite index subgroup.

(note: this only requires

$\mathcal{S} \to \mathcal{M}_{g,n}$ dominant)

Slogan

$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ MCG-finite

$\Longleftrightarrow$ universal isomonodromic deformation descends to an étale

to show

$\mathcal{S} \longrightarrow \mathcal{M}_{g,n+1}$ , i.e.,

" $\rho$ is (almost) defined on the universal curve "

## Converse to 2.1.3

Let $\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ be MCG-finite + irreducible. $\exists$ versal family of $(g,n)$ curves:

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ \Delta \arrow[u, bend left, "s_1"'] \arrow[u, bend left=60, "s_n"] \end{tikzcd}$$

,

a rep $\hat{\rho} : \pi_1(\mathcal{C}^0) \to GL_r(\mathbb{C})$ of finite order det, s.t. if $C^0$ is a fiber, then

$$\hat{\rho}\Big|_{\pi_1(C^0)} \simeq \rho$$

Considered as rep of $\pi_1(C_0)$ under $\pi_1(C_0) \simeq \pi_1(\Sigma_{g,n})$

Rmk Let $\begin{array}{c} \mathcal{D}^\circ \\ \downarrow \\ \mathcal{T}_{g,n} \end{array}$ be pullback of universal $(g,n)$ curve.

Then $\rho$ always isomonodromically deforms to a rep

$\longleftarrow$ no MCG-finite cond.'n

$$\tilde{\rho} \quad \pi_1(\mathcal{D}^\circ) \longrightarrow GL_r(\mathbb{C}).$$

The total space $\mathcal{D}^\circ$ is highly non-algebraic. If $\rho$ is MCG finite, then we may "algebraize" this: $\tilde{\rho}$ is pulled back from finite cover of $\mathcal{M}^\circ_{g,n+1}$ = universal curve over $\mathcal{M}_{g,n}$

Pf has two key parts.

☆ Given MCG-finite irreducible

$$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$$

lift it to a (unique) projective rep:

$$\tilde{\rho} : \tilde{\Gamma} \longrightarrow PGL_r(\mathbb{C}),$$

where

$$\pi_1(\Sigma_{g,n}) \subseteq \tilde{\Gamma} \underbrace{\subseteq PMCG_{g,n+1}}_{\text{finite index}}$$

$\Gamma$ $\quad \Gamma := \mathrm{Stab}_{PMCG_{g,n}}([\rho])$

$\tilde{\Gamma}$ is inverse image under

$PMCG_{g,n+1} \longrightarrow PMCG_{g,n}$ ⌡

Key: Schur's lemma

This has the following geometric corollary:

Cor Let $\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ be irreducible & MCG finite. $\exists$

- $\Delta \to \mathcal{M}_{g,n}$ finite étale, w/ family $\begin{array}{c} \mathcal{C}^0 \\ \downarrow \\ \Delta \end{array}$,
- A rep $\tilde{\rho} : \pi_1(\mathcal{C}^0) \to PGL_r(\mathbb{C})$

s.t. if $C^0$ is a fiber,

$$\tilde{\rho}\big|_{\pi_1(C^0)} \simeq \mathbb{P}(\rho) \text{ as}$$

projective reps of $\pi_1(\Sigma_{g,n})$

MCG finite representations $\rho$ of rank 1 have finite order.

Pf idea:

$$\begin{tikzcd} \rho: \pi_1(\Sigma_{g,n}) \arrow[r] \arrow[d] & \mathbb{C}^* \\ H_1(\Sigma_{g,n}) \arrow[ur] & \end{tikzcd}$$

Suppose $\rho$ has $\infty$ image. Then $\exists$ $\gamma \in H_1(\Sigma_{g,n})$ s.t. $\rho(\gamma) \notin \mu_\infty$.

Claim $\left(\mathrm{Dehn}(\tilde{\gamma})^n\right)^* \rho$ are all distinct characters.

Pf If $\sigma \in H_1(\Sigma_{g,n})$ s.t. $\sigma \cdot \gamma > 0$,

then $\left(\mathrm{Dehn}(\tilde{\gamma})^n\right)^* \rho\,(\sigma)$

$= \rho(\mathrm{Dehn}(\tilde{\gamma})^n(\sigma))$

$= \rho(\sigma + \underbrace{\boxed{\phantom{x}}}_{\text{pos, increasing w/ } n}\ \gamma)$ ✓

# Theorems

## Thm 1.2.1

For $g, n, r \geq 0$, let

$$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$$ be

MCG-finite. If

$$r < \sqrt{g+1}, \text{ then}$$

$\rho$ has finite image.

## Cor 1.3.1

Suppose $\pi^\circ : \mathscr{C}^\circ \to \mathscr{S}$ is

a versal family of $(g,n)$-curves

Let $\mathbb{V}$ be a $\mathbb{C}$-local system

on $\mathscr{C}^\circ$ of rank $< \sqrt{g+1}$.

Then for any fiber $C^o$,

$\mathbb{V}|_{C^o}$ has finite monodromy.

$=$

Three main steps to prove main theorem.

- Suppose $\rho$ is unitary, irreducible, and MCG-finite, w/ $\operatorname{rank}(\rho) < \sqrt{g+1}$

$\rightsquigarrow$ • $\exists$ $\begin{array}{c}\mathcal{C}^o \\ \downarrow \\ S\end{array}$ versal curve of type $(g, n)$

• $\tilde{\rho}: \pi_1(\mathcal{C}^o) \longrightarrow GL_r(\mathbb{C})$

s.t. $\tilde{\rho}|_{C^o} \simeq \rho$. Set $\mathbb{V}$ to be the local system.

(only uses MCG-finite

We will prove:

① $\exists$ # field $K$ s.t.
$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathcal{O}_K)$

② $\forall\ \iota : K \hookrightarrow \mathbb{C}$,
$\iota \circ \rho$ is unitary.
(a priori, only know for one such $\iota$)

To prove ①: use Thm 1.7.1
(cohomology rank bound for unitary local systems) to prove
ad $\mathbb{V}$ is cohomologically rigid
$\rightsquigarrow$ [EG] + [KP] $\Rightarrow$
$\rho$ may be defined$/\ \mathcal{O}_K$

To prove ②: cohomological rigidity $\Rightarrow$ $\iota \circ \rho$ underlies $\mathbb{C}$PVHS. As they have low rank, [LL22a] kicks in and shows they are unitary (see Bruno's talk).

- Let $\rho$ be MCG finite & semi-simple.

  ① NAHT $\Rightarrow$ deform $\mathbb{V}$ to a $\mathbb{C}$PVHS $\mathbb{V}'$

  ② By [LL22a], $\mathbb{V}'$ has unitary monodromy on any fiber $\Rightarrow$ by first step, $\mathbb{V}'$ has finite monodromy when restricted to a fiber

  ③ Use "cohomology rank bound for unitary local systems" to show:

$$\mathbb{V}'|_{C^\circ} \cong \mathbb{V}|_{C^\circ}:$$

Thm 1.7.1 will imply $\mathbb{V}'|_{C^0}$ does not admit any MCG-finite deformations.

- When $\rho$ is MCG-finite:

I don't understand, but related to Putnam-Wieland.