Introduction to algebraic flat connections

Let S/k be a smooth variety

Def A quasi-coherent sheaf w/ a connection (M, ∇) is a pair,

$$M \in QCoh(S)$$

$$\nabla : M \longrightarrow M \otimes \Omega^1_{S/k}$$

satisfying the Leibniz-rule, i.e.

$$\nabla(fm) = m \otimes df + f\nabla(m)$$

$\forall$ local sections $f \in \Gamma(U, \mathcal{O}_S)$
$m \in \Gamma(U, M)$

∇ is said to be flat if

$$\nabla \circ \nabla = 0 \quad :$$

∇ induces a Leibniz-linear map:

$$(M \otimes \Omega^1_{S/k}) \to (M \otimes \Omega^2_{S/k})$$

$$\nabla(m \otimes \omega) \mapsto \nabla m \wedge \omega + m \otimes d\omega$$

so ∇ flat $\Leftrightarrow$ $\nabla \circ \nabla : M \to M \otimes \Omega^2_{S/k}$ is identically 0.

<u>Example</u>

- S/k arbitrary smooth variety:

$M = \mathcal{O}$, $\nabla =$ "$d + \omega$" w/ $\omega \in H^0(S, \Omega^1_{S/k})$

i.e., $\nabla(f) := df + f\omega$

Exercise: when is $(\mathcal{O}, d+\omega)$ flat?

<u>Def</u> $MIC(S/k)$ = category of coherent sheaves w/ a flat connections (morphisms are horizontal)

<u>Rmk</u>: Can make this definition for any smooth morphism (of finite type).

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MOTIVATION

Let S be a smooth complex manifold. Can make an analogous def of "connection" on S.

Let (V, ∇) be a vector bundle w/ a connection. Then, for $s, t \in S$, γ a path connecting them, ∇ furnishes an iso

$$\nabla_\gamma : V_s \longrightarrow V_t$$ of

the fibers.

∇ is flat $\iff$ ∇_γ depends only on the homotopy type of γ.

★ In other words, ∇ is flat / integrable iff horizontal sections exist analytic locally.

Slogans: • A flat connection ∇ on a vector bundle V provides a canonical identification of nearby fibers of V.

• A flat connection corresponds to a system of linear differential equations on S

Fact the functor

$$\mathrm{MIC}_{C^\infty}(S) \longrightarrow \overbrace{\mathrm{Loc.Sys}}^{\text{finite rank}}(S)$$
$$(V, \nabla) \longmapsto V^{\nabla = 0}$$

is an equivalence of categories

If S is connected, and $s \in S$, we further obtain an equivalence

$$\mathrm{MIC}_{C^\infty}(S) \longrightarrow \mathrm{Rep}_{\mathbb{C}}(\pi_1(S, s))$$

The same statements are true for holomorphic manifolds.

$(M, \nabla) \rightsquigarrow H^1(S, GL_n(\mathbb{C}))$

$\Rightarrow$ a vector bundle admits a flat connection $\Leftrightarrow$ we may choose the transition functions to be constant.

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Another perspective on flat connections

Any connection (M, ∇) on S/k gives rise to a (k-linear) map of abelian sheaves:

$$T_S \longrightarrow \mathrm{End}_k(M)$$
$$\xi \longmapsto \left(m \mapsto \nabla_\xi(m) := \langle \nabla(m), \xi \rangle \right)$$

∇ is flat $\Longleftrightarrow$ this is a map of Lie algebras.

<u>Def</u> Let D_S be "the universal enveloping algebra of the Lie algebroid T_X", i.e., D_S is a (non-commutative) quasi-coherent sheaf of $\mathcal{O}_S$-algebras, generated by $\mathcal{O}_S$ & T_S, w/ the relations

- $f \cdot \partial = f\partial$
- $\partial \cdot f - f \cdot \partial = \partial(f)$

$\Big[$ $f \in \mathcal{O}_X$, $\partial \in T_X$, $\partial' \in T_X$

- $\partial\partial' - \partial'\partial = [\partial, \partial']$

$\Big[$ D_S is the ring of crystalline differential operators

Then there is a natural equivalence

$$\mathrm{MIC}_S \longrightarrow \left\{ \begin{array}{l} \text{coherent } \mathcal{O}_S \text{ modules} \\ \text{w/ an action of} \\ \mathcal{D}_S \end{array} \right\}$$

(analogous to $\mathfrak{g}$-mod $\longrightarrow$ $\mathcal{U}(\mathfrak{g})$-mod)

<u>Rmk</u> $\mathcal{D}_S$ is not the "true ring differential operators".

<u>Rmk</u> $\mathcal{D}_S$ has a filtration by "degree of differential operator":

$$\mathcal{D}_S^{\leq n} := \mathrm{Im}\Big(\bigoplus_{i \geq 0}^{n} T_S^{\otimes i} \to \mathcal{D}_S \Big)$$

$$\mathrm{gr}(\mathcal{D}_X) \simeq \mathrm{Sym}(T_X) \simeq \pi_*(\mathcal{O}_{T^*X}),$$

where $\pi : T^*X \to X$ is the cotangent bundle.

Facts about alg flat connections

- char $k=0 \Rightarrow$ if $M \in Coh(S)$ admits a flat connection, then M is a vector bundle

- <u>Exercise</u>: prove that if $S/\mathbb{C}$ is projective and $L \in Pic(S)$ admits a flat connection $\Rightarrow \deg(L)=0$

- MIC admits: $\oplus$, internal hom, $\otimes$

- $f: X' \to X \rightsquigarrow f^*: MIC_X \to MIC_{X'}$

<u>Q</u>: How do we algebraically construct flat connections? (i.e., not using RH or Simpson)

$$\begin{array}{c} \mathcal{X} \\ f \downarrow \\ S \end{array} \quad \text{smooth proj}, \; i \geq 0$$

Def: $\mathcal{H}^i_{dR}(\mathcal{X}|S) :=$

$$R^i f_* \left(\mathcal{O} \xrightarrow{d} \Omega^1_{\mathcal{X}|S} \xrightarrow{d} \Omega^2_{\mathcal{X}|S} \to \cdots \right)$$

is the ith algebraic de Rham cohomology

$\mathcal{H}$ is equipped w/ a flat connection,

∇_{GM}.

Rmk when $S|\mathbb{C}$, the corresponding local system is isomorphic to

$$R^i f_* \mathbb{C}.$$

Def Let S/k be smooth, connected variety.
A flat connection $(M, \nabla) \in \mathrm{Coh}(S)$ comes from algebraic geometry if $\exists$

- $\phi \neq \mathcal{U} \subseteq S$
- smooth proper map $\begin{matrix} \mathcal{X} \\ f \downarrow \\ \mathcal{U} \end{matrix}$
- $i \geq 0$, s.t.

$(V, \nabla)|_{\mathcal{U}}$ is a subquotient of

$$\mathcal{H}^i_{dR}(\mathcal{X}/\mathcal{U})$$

Q Let k have char 0. Which objects (M, ∇) of MIC_S come from algebraic geometry?

Last term, we discussed necessary Hodge-theoretic conditions. This term, we focus on arithmetic conditions.

<u>Conj</u> (Simpson)
Let S/k be smooth projective, w/ char $k=0$. Let $(V,\nabla) \in MIC_S$ be rigid. Then (V,∇) is of geometric origin.

One goal post of our seminar is to understand Esnault-Gröchenig's arithmetic evidence for Simpson's conjecture. To this end, we will study various flavors of non-abelian Hodge theory in char p. Time permitting, we will discuss other applications to complex geometry of these techniques.

Characteristic p

S/k smooth w/ $\mathrm{char}(k)=p$

Def

$$\begin{tikzcd} S \arrow[dr, dashed, "F_{S/k}"] \arrow[drr, bend left, "\text{finite flat}"] \arrow[ddr, bend right] & & \\ & S' \arrow[r, "\pi"] \arrow[d] & S \arrow[d] \\ & k \arrow[r, "\mathrm{Frob}_k"] & k \end{tikzcd}$$

If $S = \mathrm{Spec}\left(k[x_1,\dots,x_N] / \sum a_I x^I\right)$

Then $S' = \mathrm{Spec}\left(k[x_1, \dots, x_N] / \sum a_I^p x^I\right.$

$$\begin{tikzcd}
\mathbb{A}^N & \mathbb{A}^N \\
S \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "F_{S/k}"'] & S' \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

$$(a_1, \dots, a_n) \longmapsto (a_1^p, \dots, a_n^p)$$

<u>Observation</u>

Consider $(\mathcal{O}, d) \in MIC_X$.

Then $\mathcal{O}^{d=0}$ is big!

<u>Reason</u>: $\forall f \in \mathcal{O}, \; f^p \in \mathcal{O}^{d=0}$.

$$\left(\mathcal{O}_X^{d=0} = \mathcal{O}_{X'} \right)$$

More generally if $(E, \nabla) \in MIC_X$,

$(F_{X|k})_* E^{\nabla=0}$ is an $\mathcal{O}_{X'}$-module:

if $s \in E^{\nabla=0}$, and $f' \in \mathcal{O}_{X'}$, then

$$d(F_{X|k}^*(f')) = 0.$$

<u>Observation</u>

Given any $E \in Coh(X')$, &

can construct a canonical connection

on $F_{X|k}^*(E) := F_{X|k}^{-1}E \otimes_{F_{X|k}^{-1}\mathcal{O}_{(X')}} \mathcal{O}_X$

in MIC_X:

$$\nabla^{can}(s \otimes f) = s \otimes df$$

This is well-defined:

$$\varphi \in \mathcal{O}_{X'}, \quad \nabla^{can}(\varphi s \otimes f)$$
$$= \nabla^{can}(s \otimes f\, F_{X/k}^* \varphi)$$

$$:= s \otimes d(f\, F_{X/k}^{*}\varphi)$$

$$= s \otimes (df) \cdot F_{X/k}^{*}\varphi$$

$$= \varphi s \otimes df$$

<u>Upshot</u> Inside of MIC_X, there is a special subcategory that is obtained under Frobenius pullback.

Q: How do we intrinsically describe <u>this</u> subcategory?

p-curvature

$$T_X = \mathcal{D}er_k(\mathcal{O}_X, \mathcal{O}_X)$$

Let ∂ be a local section of T_X, a.k.a. a derivation.

Claim: $\underbrace{\partial \circ \cdots \circ \partial}_{p}$ is also a derivation!

Pf $\partial^N(fg) = \sum_{i=0}^{N} \binom{N}{i} \partial^i f \, \partial^{N-i} g$

Set $\partial^{[p]} := \partial \circ \cdots \circ \partial$ as an element of T_X.

Def The p-curvature is the map of sheaves

$$\begin{aligned} T_X &\longrightarrow \mathcal{D}_X^{\leq p} \subseteq \mathcal{D}_X \\ \partial &\longmapsto \partial^p - \partial^{[p]} \end{aligned}$$

Let $(E,\nabla) \in MIC_X$. Then E has the structure of a $\mathcal{D}_X$-module.

The p-curvature is the (a priori k-linear map):

$$T_X \longrightarrow \mathcal{E}nd_k(E)$$

$$\partial \longmapsto \text{"}\partial^p - \partial^{[p]} \text{ on } E\text{"}$$

Observation

$(F^*_{X/k}(E), \nabla^{can})$ has p-curvature 0.

Pf Must compute $\nabla^{can}_\partial \circ \cdots \circ \nabla^{can}_\partial - \nabla^{can}_{\partial \circ \cdots \circ \partial}$ for $\partial \in T_X$.

$$\nabla^{can}_\partial \circ \cdots \circ \nabla^{can}_\partial (s \otimes f) = s \otimes \partial^p f$$

$$\nabla^{can}_{\partial \circ \cdots \circ \partial} (s \otimes f) = s \otimes \partial^p f$$

(Comes down to p-curvature of $(\mathcal{O}, d)$ being 0)

Thm (Cartier Descent)

(E, ∇) is a Frobenius pullback iff its p-curvature is 0.