The arithmetic of vector bundles with a flat connection

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1 Introduction

Definition 1.1. *Let X/k be a smooth variety over a field and let $M \in \mathsf{QCoh}(X)$ be a quasi-coherent sheaf on X. A* connection on M *is a map*

$$\nabla \colon M \to M \otimes \Omega^1_{X/k}$$

that satisfies the Leibniz property: for all open subsets $U \subset X$, and all local sections $f \in \Gamma(U, \mathcal{O}_X), s \in \Gamma(U, M)$, we have $\nabla(fs) = f\nabla(s) + s \wedge df$. A connection ∇ is said to be flat (or integrable) *if $\nabla \circ \nabla = 0$.*

Given a smooth variety X/k, we denote $\mathbf{MIC}(X)$ the category of quasi-coherent sheaves with an integrable connection. When $\mathrm{char}(k) = 0$, any coherent sheaf M that admits a connection is automatically locally free, i.e., is a vector bundle. We note that the notion of a quasi-coherent sheaf with a relative integrable connection may be defined over any smooth morphism $\mathcal{X} \to S$, by using $\Omega^1_{\mathcal{X}/S}$ in Definition 1.1.

The link to the seminar last semester is the following.

Fact 1.2. *Let $X/\mathbb{C}$ be a smooth projective variety. The functor from the category of vector bundles with a flat connection on X to the category of finite dimensional $\mathbb{C}$-local systems on X^{an} given by the formula on sheaves:*

$$(M, \nabla) \longmapsto M^{\nabla=0},$$

is an equivalence of categories.

One may extend this equivalence to arbitrary smooth varieties using Deligne's work on the Riemann-Hilbert problem [Del70].

Following last semester, we make the following definition.

Definition 1.3. *Let $X/\mathbb{C}$ be a smooth, connected, projective variety. Let (M, ∇) be a vector bundle with a flat connection on X. We say that (M, ∇) is* of geometric origin *(or geometric, or motivic) if threre exists an open dense $U \subset X$, a smooth projective morphism $f \colon Y \to U$, and an integer $i \geq 0$ such that $L|_U$ is a subquotient of:*

$$(\mathcal{H}^i_{dR}(Y/U), \nabla_{GM}),$$

where $\mathcal{H}^i(Y/U)$ stands for the relative de Rham cohomology and ∇_{GM} stands for the Gauss-Manin *connection.*

Recall that local systems only depend on the topology of X^{an} (and in particlar not on the algebraic structure). On the other hand, the notion of a vector bundle with a flat connection manifestly does depend on the complex/algebraic structure.

Last semester we discussed several conjecturally sufficient conditions to guarantee a complex local system is of geometric origin.[1] This term we will focus on analogous questions about vector bundles with a flat connection (the *de Rham* side). In particular, we will focus on the notion of flat connections in positive/mixed characteristic. In the context of positive characteristic, a new player emerges: *the p-curvature*. Briefly, the p-curvature emerges from the fact that in positive characteristic, $M^{\nabla=0}$ actually defines a coherent sheaf on the Frobenius twist X' and hence is very far from being locally constant. When the p-curvature is zero, we say that (M, ∇) has a *complete set of algebraic solutions*

We have the following orienting conjecture.

[1] For example, that the local system underlies a $\mathbb{Z}$-PVHS, or that there exists an isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$ such that the associated $\bar{\mathbb{Q}}_\ell$-topological local system extends to an étale local system and is moreover arithmetic.

Conjecture 1.4 (General Grothendieck-Katz p-curvature conjecture)**.** *Let $\mathbb{Z} \subset A \subset \mathbb{C}$ be an integral domain of finite type over $\mathbb{Z}$ and let $S = \mathsf{Spec}(A)$. Let $\mathcal{X}/S$ be a smooth projective morphism, and write $X_{\mathbb{C}}$ for the pullback to the point $\mathsf{Spec}(\mathbb{C}) \to S$.*

Let (M, ∇) be a vector bundle with flat connection on $\mathcal{X}/S$. Then:

1. *The flat connection $(M, \nabla)|_{X_{\mathbb{C}}}$ has finite monodromy if and only if for all $p \gg 0$ and for all closed points s of characteristic p, the p-curvature of $(M, \nabla)|_{\mathcal{X}_s}$ is 0.*
2. *Assume further that (M, ∇) is irreducible. The flat connection $(M, \nabla)|_{X_{\mathbb{C}}}$ is of geometric origin if and only if for all $p \gg 0$ and for all closed points s of characteristic p, the p-curvature of $(M, \nabla)|_{\mathcal{X}_s}$ is nilpotent.*

The first part Conjecture 1.4 is what is known as the Grothendieck-Katz p-curvature conjecture. The second part has been suggested by many mathematicians, including Bombieri, Dwork, Esnault, and Ogus.

On the other hand, we have the following conjecture of Simpson.

Conjecture 1.5. *Let $X/\mathbb{C}$ be a smooth, projective, connected variety. Let (M, ∇) be an irreducible rigid local system with torsion determinant on X. Then (M, ∇) is of geometric origin.*

Recent work of Esnault-Gröchenig has revealed a remarkable interplay between Conjecture 1.4 and Conjecture 1.5 [EG20]. Understanding portions of this work will be one of the main goals for this semester.

2 Topics

We aim to discuss the following topics.

- The definition and basic properties of the p-curvature.
- The crystalline site on a smooth variety X/k over a perfect field. The notion of crystals and isocrystals on this site.
- Nonabelian Hodge theory in positive characteristic, following Lan-Sheng-Zuo (and having origins in Ogus-Vologodsky). The notion of a periodic Higgs-de Rham flow in characteristic p, due to Lan-Sheng-Zuo. Main reference: [LSZ15, LSZ21]. Secondary reference: [OV07]
- The Hitchin fibration for the *de Rham* moduli space on a curve X/k over a perfect field of positive characteristic. The correspondence between the Higgs and de Rham moduli spaces over the Hitchin base, again in the context of a curve. Main reference: [G15]. Secondary references: [BB07, BMR08]. (Realistically, we will only have careful statements here.)
- Two proofs (one due to Esnault-Gröchenig, the other due to Esnault-de Jong) of the following result. Let $S = \mathsf{Spec}(A)$, with $A \subset \mathbb{C}$ an integral domain of finite type over $\mathbb{Z}$, and let $\mathcal{X}/S$ be a smooth projective family. Let $(\mathcal{M}, \nabla)$ be a flat connection on $\mathcal{X}/S$. Suppose $(\mathcal{M}, \nabla)|_{X_{\mathbb{C}}}$ is cohomologically rigid. Then for all $p \gg 0$, for all closed points s of residue characteristic p, completing at s yields an F-isocrystal on $\mathcal{X}_s$.
- Time permitting, applications of non-abelian Hodge theory in positive characteristic to questions in complex geometry, after Arapura and Langer.

3 Outline

1. (3 lectures) **Goals**: Introduction to flat connections, the ring of crystalline differential operators, the definition and basic properties of p-curvature, Cartier Descent, Katz/Deligne's theorem that the p-curvature of a family coming from geometry is nilpotent. Explain the relation to crystals/isocrystals on the crystalline and nilpotent crystalline site. Time permitting, present Mochizuki's perspective on p-curvature.

2. (2-3 lectures) **Goals**: Explain mod p non-abelian Hodge theory via exponential twisting.

   State the main theorem of [LSZ15]. Carefully explain the key-input: [LSZ15, Lemma 2.1] (including the background material from Deligne-Illusie). Explain the functors in [LSZ15, Sections 2.2 and 2.3].

   Explain [LSZ15, Lemma 4.1], which specifies the inverse Cartier in the case of a geometric family. Give the definition of a Higgs-de Rham flow and a periodic Higgs-de Rham flow [LSZ21] (in the case of 1-periodicity, this appears as a "stationary point" in [Ara17]).

3. (2 lectures) **Goal**: Sketch the BNR correspondence, [EG20, Theorem 2.17] or [G15, Proposition 3.15].

   Sketch the proof of the above theorem, including background material from [G15]. This will require: defining the *de Rham Hitchin fibration* as in [EG20, Section 2.5], stating the BNR correspondence [EG20, Theorem 2.17], and giving an indication of the proof (either written up in [EG20] or [G15]). Explain why the de Rham Hitchin fibration is proper. ([Lan14] might also be useful.)

4. (1 Lecture) **Goal**: Prove that stable rigid flat connections are globally nilpotent.

   Explain [EG20, Lemma 3.4, Proposition 3.5]. Prove [EG20, Theorem 1.4]

5. (2 lectures) **Goal**: explain the key counting argument in [EG20, Lemma 4.11], i.e., indicate why stable rigid flat connections yield, on reduction modulo p, periodic Higgs-de Rham flows.

   Explain enough of [EG20, Section 4.2] to prove [EG20, Lemma 4.11]. Give an indication as to what the p-adic periodicity statement is (without defining the p-adic inverse Cartier transform). State the corollary that stable rigid flat connections yield F-crystals (on the crystalline site). Another useful reference might be [E22, Section 9]

6. (1 lecture) **Goal**: explain [E22, Theorem 8.4], i.e., the alternative argument of Esnault-de Jong that stable rigid flat connections yield F-isocrystals.

7. (remaining lectures?) Explain one of the theorems [Ara17, Lan15, Lan16].

References

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