Non-abelian Hodge theory mod p, I

Notation:

p prime integer

k perfect field of char p

$\phi : k \to k$ absolute Frobenius of k.

If $X \to \operatorname{Spec} k$ morphism

$F_X : X \to X$ absolute Frobenius

$F = F_{X/k} : X \to X'$ relative Frobenius

1. The Lemma of Deligne-Illusie

$$f : X \to Y \in Sch \text{ is smooth}$$

if

- f is locally of finite presentation

$$\begin{tikzcd} & & X \arrow[d, "f \text{ smooth}"] \\ T_0 \arrow[urr, bend left, "g_0"] \arrow[r, hook, "i", "\text{square-zero}"'] & T \arrow[ur, dashed, "\text{locally on } T"] \arrow[r] & Y \end{tikzcd} \qquad (*)$$

→ When does there exist a global extension of g_0?

<u>Proposition</u> 1.1.1: Situation as in (*), I ideal of i.

1. $\exists$ obstruction $c(g_0) \in Ext^1(g_0^* \Omega^1_{X/Y}, I)$ s.t.

$$c(g_0) = 0 \iff \exists \text{ global extension}$$

2. if $c(g_0) = 0$ then $\{\text{global extensions}\}$ is an affine space under $Hom(g_0^* \Omega^1_{X/Y}, I)$

<u>Proof</u>: f smooth $\Rightarrow \Omega^1_{X/Y}$ finite loc. free

$\Rightarrow \mathcal{H}om(g_0^* \Omega^1_{X/Y}, I) = g_0^* T_{X/Y} \otimes I =: G$

$\Rightarrow Ext^1(g_0^* \Omega^1_{X/Y}, I) = H^1(T_0, g_0^* T_{X/Y} \otimes I)$

Let $E \in Sh(T_0)$:

$$U_0 \mapsto \{ g^b \mid g \in Hom_Y(U, X), g_0 = g \circ i \}$$

$(g^b : \mathcal{O}_X \to g_* \mathcal{O}_U)$.

U open subscheme corresponding to U_0.

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$\rightsquigarrow E$ is the sheaf of all

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d, hook] & X \arrow[d] \\ U \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

<u>Sub Lemma 1.1.2</u>

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d, "f"] \arrow[dl, dashed] \\ B \arrow[r] & B/I \end{tikzcd} \in \mathrm{Ring}$$

$I^2 = 0$.

1. ϕ_1, ϕ_2 lifts of $f \Rightarrow \delta = \phi_1 - \phi_2 : A \to I$ R-lin. der.
2. ϕ lift, $\delta : A \to I$ R-lin. der $\Rightarrow \phi + \delta$ lift of f.

<u>Proof</u>: Exercise.

$g^b \in E(U_0)$, $\delta \in G(U_0)$

use adjunction + comp to define

$$g^b + \delta \in E(U_0).$$

$\xRightarrow[\text{+ smoothness of } f]{\text{Sublemma (1)}}$ E is a G-torsor

$\Longrightarrow c(E) \in H^1(T_0, G)$

$c(E) = 0 \iff E \simeq G$ as G-sheaves

$\iff E(T_0) \neq \emptyset$

in that case $E(T_0)$ is an affine space under

$G(T_0) = \mathrm{Hom}(g_0^* \Omega^1_{X/Y}, I)$. $\square$

$i : Y_0 \hookrightarrow Y$ thick of order 1

$g_0 : X_0 \to Y_0$ flat (smooth)

<u>lift of X_0 over Y</u> is $X \to Y$ s.t.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y_0 \arrow[r] & Y \end{tikzcd}$$

morphism is commutative diagram.

Example 1.13

$W_2(k)$ ring of Witt vectors of length 2

- as sets $W_2(k) = k^2$
- $W_2(k) \longrightarrow k$ surjective
- $W_2(\mathbb{F}_p) = \mathbb{Z}/p^2\mathbb{Z}$

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} W_2(k) \arrow[d, "\text{flat} \Leftrightarrow k \text{ perfect}"] \\
\operatorname{Spec} \mathbb{F}_p \arrow[r, hook, "\text{thickening of order } 1"'] & \operatorname{Spec}(\mathbb{Z}/p^2\mathbb{Z})
\end{tikzcd}$$

$\overset{\text{flatness}}{\Longrightarrow}$ $\ker(W_2(k) \longrightarrow k) = p\, W_2(k)$

$\Longrightarrow$ $\ker(W_2(k) \xrightarrow{\cdot p} W_2(k)) = p W_2(k)$

$\Longrightarrow$ $\underline{p} : k \longrightarrow p\, W_2(k)$ induced by mult. with p!

Remark: $\exists\, \omega(g_0) \in \operatorname{Ext}^2(\Omega^1_{X_0/Y_0}, g_0^* I)$ if g_0 smooth

s.t. $\omega(g_0) = 0 \iff \exists$ smooth lift of X_0 over Y.

1.2 Lifting the relative Frobenius

$X \to \operatorname{Spec} k$

$$\Longrightarrow \quad F^* \Omega^1_{X'/k} \longrightarrow \Omega^1_{X/k} \text{ is zero}$$

$$\begin{tikzcd} A' \arrow[r, "F_{A/k}"] & A \\ k \arrow[u] \arrow[r, "id"] & k \arrow[u] \end{tikzcd}$$

$$d F_{A/k}(a \otimes \lambda) = d(a^p \lambda) = \lambda\, d a^p$$
$$= \lambda p a^{p-1} da = 0.$$

Notation: $X \to k$ smooth

$\tilde{X}$ smooth lift of X over $W_2(k)$

$\tilde{X}'$ " " X' "

$\tilde{F}: \tilde{X} \to \tilde{X}'$ lift of F

$$\rightsquigarrow \quad \begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & X' \arrow[d, "\pi'"] \\ \tilde{X} \arrow[r, "\tilde{F}"] & \tilde{X}' \end{tikzcd} \quad \text{commutes.}$$

Remark: $\phi: k \to k$ is

lifting X to $W_2(k) \longleftrightarrow$ lifting X' to $W_2(k)$

- obstruction to lift X to $W_2(k)$

$$\in Ext^2(\Omega^1_{X/k}, \mathcal{O}_X) \simeq H^2(X, T_{X/k})$$

- obstruction to lift $F: X \to X'$

$$\in Ext^1(F^*\Omega^1_{X'/k}, \mathcal{O}_X) \simeq H^1(X', T_{X/k} \otimes F_*\mathcal{O}_X)$$

$\Longrightarrow$ both vanish locally!

Lemma 1.2.4:

1. multiplication by p induces iso

$$\underline{p}: \Omega^1_{X/k} \xrightarrow{\sim} p\,\Omega^1_{\tilde{X}/W_2(k)}$$

2.

$$\begin{tikzcd} \Omega^1_{\tilde{X}'/W_2(k)} \arrow[r] \arrow[dr, dashed] & \tilde{F}_*\Omega^1_{\tilde{X}/W_2(k)} \\ & p\,\tilde{F}_*\Omega^1_{\tilde{X}/W_2(k)} \arrow[u, hook] \end{tikzcd}$$

<u>Proof</u>

1. Example 1.1.3 + base-change formula for Ω^1 (need smoothness!)

2.
$$\begin{array}{ccc}
\Omega^1_{\tilde{X}'/W_2(k)} & \longrightarrow & \tilde{F}_* \Omega^1_{\tilde{X}/W_2(k)} \\
 & \updownarrow & \\
F^* \Omega^1_{\tilde{X}'/W_2(k)} & \longrightarrow & \Omega^1_{\tilde{X}/W_2(k)}
\end{array}$$

π closed immersion $\Longrightarrow$

$$\begin{tikzcd}
\Omega^1_{\tilde{X}'/W_2(k)} \arrow[r] \arrow[dr, dashed] & \tilde{F}_* \Omega^1_{\tilde{X}/W_2(k)} \\
 & p\tilde{F}_* \Omega^1_{\tilde{X}/W_2(k)} \arrow[u, hook]
\end{tikzcd}$$

$$\Updownarrow$$

$$\begin{tikzcd}
\pi^* \tilde{F}^* \Omega^1_{\tilde{X}'/W_2(k)} \arrow[r] \arrow[d, "\simeq"'] & \pi^* \Omega^1_{\tilde{X}/W_2(k)} \arrow[d, "\simeq"] \\
F^* \Omega^1_{X'/k} \arrow[r, "0"] & \Omega^1_{X/k}
\end{tikzcd}$$
is zero

$\square$

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Lemma 1.2.4

$\Longrightarrow$

$$
\begin{tikzcd}
\Omega^1_{\tilde X/W_2(k)} \arrow[r] \arrow[d, "\simeq"'] & p\,\tilde F_* \Omega^1_{\tilde X/W_2(k)} & \\
\pi'_* \Omega^1_{X'/k} \arrow[r, dashed, "\exists!"] & \tilde F_* \pi_* \Omega^1_{X/k} \arrow[u, "\simeq", "p"'] \arrow[r, "\simeq"] & \pi'_* F_* \Omega^1_{X/k}
\end{tikzcd}
$$

$$\exists!\ \varphi_{\tilde F} : \Omega^1_{X'/k} \to F_* \Omega^1_{X/k}$$

"division by p"

What does $\varphi_{\tilde F}$ do?:

$$
\begin{tikzcd}
a' \arrow[rrr, mapsto, bend left] & A' \arrow[r, "F"] & A & a^p \quad a \\
\tilde a' \arrow[u, mapsto] & \tilde A' \arrow[u] \arrow[r, "\tilde F"] & \tilde A \arrow[u] & \boxed{\tilde a} \arrow[u, mapsto]
\end{tikzcd}
$$

$\Longrightarrow$ $F(\tilde a') \equiv a^p \bmod p$

$\Longrightarrow$ $\tilde F(\tilde a) = \tilde a^p + p\tilde b$

$\Longrightarrow$ $d\tilde F(d\tilde a') = p\tilde a^{p-1} d\tilde a + p\, d\tilde b$

$$\boxed{\varphi_{\tilde F}(d\tilde a) = a^{p-1}\,da + db}$$

1.3 The Lemma of Deligne-Illusie

Lemma 1.3.1 (D-I.) Let $(\tilde F_1 : \tilde X_1 \to \tilde X', \tilde F_2 : \tilde X_2 \to \tilde X')$ be a pair of lifts of F.

$\exists$ canonical $h(\tilde F_1, \tilde F_2) : \Omega^1_{X'/k} \to F_* \mathcal{O}_X$

s.t.

1. $\varphi_{\tilde F_2} - \varphi_{\tilde F_1} = d\, h(\tilde F_1, \tilde F_2)$.
2. if $\tilde F_3 : \tilde X_3 \to \tilde X'$ a third lift

$$\Longrightarrow h(\tilde F_1, \tilde F_2) + h(\tilde F_2, \tilde F_3) = h(\tilde F_1, \tilde F_3).$$

Proof: Assume $\tilde F_1$ and $\tilde F_2$ are isomorphic via u

$$\Longrightarrow \begin{tikzcd} & & \tilde X' \arrow[d] \\ X \arrow[r, hook] \arrow[urr, bend left, "\pi' \circ F"] & \tilde X_1 \arrow[ur, dashed, shift left, "\tilde F_1"] \arrow[ur, dashed, shift right, "\tilde F_2 \circ u"'] \arrow[r] & \mathrm{Spec}(W_2(k)) \end{tikzcd}$$

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$$\overset{1.1.1}{\Longrightarrow} \exists\, h_u \in \mathrm{Hom}(F^*\Omega^1_{X'/k}, \mathcal{O}_X) \quad \text{s.t.}$$

$$F_1^\flat - (\tilde{F}_2 \circ u)^\flat = h_u$$

Suppose v is a second iso, then

$$\begin{tikzcd}
 & & \tilde{X}_2 \arrow[d] \\
X \arrow[r] \arrow[urr, bend left, "\pi_2"] & \tilde{X}_1 \arrow[ur, dashed, shift left, "u"] \arrow[ur, dashed, shift right, "v"'] \arrow[r] & \mathrm{Spec}(W_2(k))
\end{tikzcd}$$

$$\overset{1.1.1}{\Longrightarrow} \alpha \in \mathrm{Hom}(\Omega^1_{X/k}, \mathcal{O}_X) \quad \text{s.t.}$$

$$u^\flat - v^\flat = \alpha.$$

$$\Longrightarrow (\tilde{F}_1 \circ u)^\flat - (\tilde{F}_2 \circ v)^\flat = \alpha \circ (F^*\Omega^1_{X'/k} \to \Omega^1_{X/k})$$

$$= 0$$

$$\Longrightarrow h_u = h_v$$

$\tilde{F}_3 : \tilde{X}_3 \to \tilde{X}'$ third lift, isos

$$u : \tilde{F}_1 \simeq \tilde{F}_2$$
$$v : \tilde{F}_2 \simeq \tilde{F}_3$$
$$w : \tilde{F}_1 \simeq \tilde{F}_3$$

$$h_W = h_{V\cap U} = h_V + h_U$$

$$\varphi_{\tilde{F}_1} - \varphi_{\tilde{F}_2} = d\, h_U$$

follows from explicit description of $\varphi_{\tilde{F}_1}$.

$\xRightarrow{1.1.1}$ $\tilde{X}_1$ — $\tilde{X}_2$ locally (H^1 vanishes for affine)

$$\Rightarrow h(\tilde{F}_1, \tilde{F}_2) := h_U$$

where U s.t.
$$\begin{tikzcd} \tilde{X}_1|_U \arrow[rr, "\simeq"] \arrow[dr] & & \tilde{X}_{2U} \arrow[dl] \\ & U & \end{tikzcd}$$ $\square$

NAHT mod p, II ①

Goal: Construct a variant of $C_{\tilde{X}}$ and $C_{\tilde{X}}^{-1}$ due to LSZ $C_{\exp,\tilde{X}}$ and $C_{\exp,\tilde{X}}^{-1}$ (exponential twisting)

Choose and fix $W_2(k)$-lift $\tilde{X}$ of X

$\tilde{X}'$ of X'

(no extra datum since k is perfect)

Assume: $\exists$ global Frobenius lift

$\tilde{F} : \tilde{X} \to \tilde{X}'$ of F

Afterwards explain gluing.

2.2 The inverse Cartier transform

$$(E, \Theta) \in HIG_{\leq p}(X'/k)$$

$$\rightsquigarrow \text{want } C_{\exp,\tilde{X}}^{-1}(E,\Theta) \in MIC_{\leq p}(X/k)$$

Define

$$M := F^* E$$

$$\nabla := \nabla^{can} + \varphi_{\tilde{F}}(F^*\Theta)$$

What is $\varphi_{\tilde{F}}(F^*\Theta)$?

$$\begin{tikzcd} F^*E \arrow[r, "F^*\Theta"] \arrow[rr, bend right, "\mathcal{O}_X\text{- linear}"'] & F^*E \otimes F^*\Omega^1_{X'/k} \arrow[r, "1\otimes\varphi_{\tilde{F}}"] & F^*E \otimes \Omega^1_{X/k} \end{tikzcd}$$

$\Longrightarrow \varphi_{\tilde{F}}(F^*\Theta) \in \mathcal{E}nd_{\mathcal{O}_X}(F^*E) \otimes \Omega^1_{X/k}$

$\Longrightarrow \nabla$ is a connection on M

<u>∇ is flat</u>

<u>Lemma 2.2.1</u> Let ∇ be (any) connection on a cohert sheaf N. There exists a unique family of k - linear maps

$$d^\nabla : N \otimes \Omega^i_{X/k} \longrightarrow N \otimes \Omega^{i+1}_{X/k}$$

s.t.

1. $(d^\nabla)_0 = \nabla$
2. $\alpha \in \Omega^m_{X/k}, \beta \in M \otimes \Omega^n_{X/k}$

$$d^{\nabla}(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^m \alpha \wedge d^{\nabla}\beta.$$

<u>Lemma 2.2.2</u>: Let N be a coherent sheaf on X, ∇ a connection on N, $\alpha \in \mathrm{End}_{\mathcal{O}_X} N \otimes \Omega^1_{X/k}$. Then for $\nabla' = \nabla + \alpha$ we have

$$K(\nabla') = K(\nabla) + \alpha \wedge \alpha + d^{\nabla}\alpha.$$

To prove ∇ is flat, have to show

1. $\varphi_{\tilde{F}}(F^*\Theta) \wedge \varphi_{\tilde{F}}(F^*\Theta) = 0$
2. $d^{\nabla^{can}}\varphi_{\tilde{F}}(F^*\Theta) = 0$

—

1. Follows from $\Theta \wedge \Theta = 0$.
2. We can check locally, assume $\Theta = v \otimes \omega$.

$$d^{\nabla^{can}}(\varphi_{\tilde{F}}(F^*\Theta)) = d^{\nabla^{can}}\varphi_{\tilde{F}}(F^*(v \otimes \omega))$$

$$= d^{\nabla^{can}}\varphi_{\tilde{F}}(F^*v \otimes F^*\omega)$$

$$= d^{\nabla^{can}}(F^*v \otimes \varphi_{\tilde{F}}(F^*\omega))$$

$$= -\nabla^{can}(F^*v) \wedge \varphi_{\tilde{F}}(F^*\omega) + F^*v \otimes d\varphi_{\tilde{F}}(F^*\omega)$$

For $s \otimes f \in F^*E = F^{-1}E \otimes_{F^{-1}\mathcal{O}_{X'}} \mathcal{O}_X$

have

$$\left(\nabla^{can}(F^*\nu)\right)(s \otimes f)$$
$$= \nabla^{can}\left(F^*\nu(s \otimes f)\right) - F^*\nu\left(\nabla^{can}(s \otimes f)\right)$$
$$= \nabla^{can}\left(\nu(s) \otimes f\right) - F^*\nu(s \otimes df)$$
$$= \nu(s) \otimes df - \nu(s) \otimes df$$
$$= 0.$$

this is not correct

To show $d\varphi_{\tilde{F}}(F^*\omega) = 0$ assume $X = \operatorname{Spec} A$, $X' = \operatorname{Spec} A'$. Let $a \in A$ and $a' \in A'$ with $F(a') = a^p$. Then

$$d\varphi_{\tilde{F}}(da') = d\left(a^{p-1}da + db\right)$$
$$= (p-1)\, a^{p-2}\, da \wedge da$$
$$= 0.$$

$$\Longrightarrow \nabla \text{ is flat}$$

<u>Claim</u>: The p-curvature of ∇ is $F^*\Theta$!!

$$\Longrightarrow (M, \nabla) \in MIC_{sp}(X/k).$$

2.3 The Cartier transform

$(M, \nabla) \in MIC_{\leq p}(X/k)$

$\psi_\nabla : M \to M \otimes F^*\Omega^1_{X'/k}$

p-curvature of ∇

Notice

$$\begin{tikzcd} M \arrow[r, "\psi_\nabla"] \arrow[rr, bend right, "\mathcal{O}_X\text{-linear}"'] & M \otimes F^*\Omega^1_{X'/k} \arrow[r, "1\otimes\varphi_{\tilde{F}}"] & M \otimes \Omega^1_{X/k} \end{tikzcd}$$

$\Rightarrow \varphi_{\tilde{F}}(\psi_\nabla) \in End_{\mathcal{O}_X}(M) \otimes \Omega^1_{X/k}$

$\Rightarrow \nabla' = \nabla + \varphi_{\tilde{F}}(\psi_\nabla)$ is a connection on M.

∇' is flat:

Since $\psi_\nabla \wedge \psi_\nabla = 0$, so suffices to show

$d^\nabla \varphi_{\tilde{F}}(\psi_\nabla) = 0.$

Locally $\psi_\nabla = \nu \otimes F^*\omega$.

$$d^\nabla \varphi_{\tilde{F}}(\psi_\nabla) = d^\nabla(\nu \otimes \varphi_{\tilde{F}}(F^*\omega))$$

$$= -\nabla\nu \wedge \varphi_{\tilde{F}}(F^*\omega) + \nu \otimes d\varphi_{\tilde{F}}(F^*\omega)$$

$$= -\nabla\nu \wedge \varphi_{\tilde{F}}(F^*\omega)$$

Since ψ_∇ is parallel with respect to $\nabla \otimes \nabla^{can}$.

$$\Longrightarrow \nabla\nu \otimes F^*\omega + \nu \otimes \nabla^{can} F^*\omega = 0$$

Q. ! How to conclude that $\nabla\nu \wedge \varphi_{\tilde{F}}(F^*\omega) = 0$

Furthermore ψ_∇ is parallel with respect to ∇' and $\nabla' \otimes \nabla^{can}$. !!

Claim (LSZ): (M, ∇') has vanishing p-curvature !!!

Cartier descent $\Longrightarrow \exists\ E \in \mathrm{Coh}(X')$ s.t.

$$(M, \nabla') \simeq (F^*E, \nabla^{can})$$

$$\Longrightarrow (M \otimes F^*\Omega^1_{X'/k}, \nabla \otimes \nabla^{can})$$

has vanishing p-curvature

$$\Rightarrow (F^*(E \otimes \Omega^1_{X'/k}), \nabla^{can}) \simeq (M \otimes F^*\Omega^1_{X'/k}, \nabla \otimes \nabla^{can})$$

Since $\psi_\nabla : M \to M \otimes F^*\Omega^1_{X'/k}$ is parallel

$$\exists\ \Theta : E \to E \otimes \Omega^1_{X'/k}\ \ \mathcal{O}_{X'}\text{-linear}$$

with

$$F^*\Theta = \psi_\nabla$$

$$\Rightarrow \quad \Theta \wedge \Theta = 0$$

$$\Rightarrow \quad \Theta \text{ nilpotent of exponent} < p.$$

2.4 Gluing

k perfect $\Rightarrow$ can find open covers $(\tilde{U}_\alpha)$ of $\tilde{X}$ and $(\tilde{U}'_\alpha)$ of X s.t. for $U_\alpha = \tilde{U}_\alpha \times_{W_2(k)} k$, $U'_\alpha = \tilde{U}'_\alpha \times_{W_2(k)} k$

$$\begin{tikzcd} U'_\alpha \arrow[r, dash] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U_\alpha \arrow[d] \\ k \arrow[r, "F_k"] & k \end{tikzcd}$$

$\forall \alpha$ choose lift

$$\tilde{F}_\alpha : \tilde{\mathcal{U}}_\alpha \to \tilde{\mathcal{U}}'_\alpha$$

of

$$F_\alpha : \mathcal{U}_\alpha \to \mathcal{U}'_\alpha$$

$\forall \alpha$ get "division by p"

$$\varphi_\alpha := \varphi_{\tilde{F}_\alpha} : F_\alpha^* \Omega^1_{\mathcal{U}'_\alpha/k} \to \Omega^1_{\mathcal{U}_\alpha/k}$$

<u>Lemma 2.4.1</u> There exist morphisms

$$h_{\alpha\beta} : F_{\alpha\beta}^* \Omega^1_{\mathcal{U}'_{\alpha\beta}/k} \to \mathcal{O}_{\mathcal{U}_{\alpha\beta}}$$

s.t.

1) $\varphi_\alpha - \varphi_\beta = d\, h_{\alpha\beta}$ over $\mathcal{U}_{\alpha\beta}$
2) $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$ over $\mathcal{U}_{\alpha\beta\gamma}$.

<u>Proof</u>:

$$\begin{tikzcd} \tilde{Z}_\alpha \arrow[r, "\tilde{G}_\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathcal{U}}'_{\alpha\beta} \arrow[d] \\ \tilde{\mathcal{U}}_\alpha \arrow[r, "\tilde{F}_\alpha"] & \tilde{\mathcal{U}}'_\alpha \end{tikzcd}$$

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applying $-\times_{W_2(k)} k$ gives

$$\begin{tikzcd} Z_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U}'_{\alpha\beta} \arrow[d] \\ \mathcal{U}_\alpha \arrow[r, "F_\alpha"] & \mathcal{U}'_\alpha \end{tikzcd} \quad (*)$$

$\Rightarrow Z_\alpha = \mathcal{U}_{\alpha\beta}$ and $(*)$ is

$$\begin{tikzcd} \mathcal{U}_{\alpha\beta} \arrow[r, "F_{\alpha\beta}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U}'_{\alpha\beta} \arrow[d] \\ \mathcal{U}_\alpha \arrow[r, "F_\alpha"] & \mathcal{U}'_\alpha \end{tikzcd}$$

$\Rightarrow \tilde{G}_\alpha : \tilde{Z}_\alpha \to \tilde{\mathcal{U}}'_{\alpha\beta}$ is a lift of $F_{\alpha\beta}$.

Similarly get $\tilde{G}_\beta : \tilde{Z}_\beta \to \tilde{\mathcal{U}}'_{\alpha\beta}$ lift of $F_{\alpha\beta}$

Now apply D-I to

$$(\tilde{G}_\alpha : \tilde{Z}_\alpha \to \tilde{\mathcal{U}}'_{\alpha\beta}, \tilde{G}_\beta : \tilde{Z}_\beta \to \tilde{\mathcal{U}}'_{\alpha\beta}) \quad \square.$$

For $C^{-1}_{\exp,\tilde{x}}$ glue local data with

$$\exp(h_{\alpha\beta}(F^*\Theta)) := \sum_{i=0}^{p-1} \frac{(h_{\alpha\beta}(F^*\Theta))^i}{i!}$$

For $C_{\exp,\tilde{x}}$ glue local data with

$$\exp(h_{\alpha\beta}(\psi_F)) = \sum_{i=0}^{p-1} \frac{(h_{\alpha\beta}(\psi))^i}{i!}.$$