CONNECTIONS, CURVATURE, AND p-CURVATURE

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1. Classical theory

We begin by describing the classical point of view on connections, their curvature, and p-curvature, in terms of maps of sheaves on a scheme.

1.1. **Connections and derivations.** Let S be a scheme, and X smooth and finite type over S throughout. We have the rank-n vector bundle $\Omega^1_{X/S}$ of 1-forms on X over S. We also fix some notation in the case that S is of characteristic p: in this case, we denote by F_T the absolute Frobenius map for any scheme T, by $X^{(p)} = X \times_S S$ the p-twist of X over S, where the map used in the fiber product is $F_S : S \to S$, and $F : X \to X^{(p)}$ the relative Frobenius map. We will also consider $\mathscr{O}_{X^{(p)}}$ as a sub-module of $\mathscr{O}_X$, identified with the kernel of $d : \mathscr{O}_X \to \Omega^1_{X/S}$.

Definition 1.1. Given any coherent sheaf $\mathscr{E}$ on X, a **connection** (called S-connection in case of ambiguity) on $\mathscr{E}$ is an $\mathscr{O}_S$-linear map $\nabla : \mathscr{E} \to \Omega^1_{X/S} \otimes \mathscr{E}$ satisfying the connection rule

$$\nabla(fs) = f\nabla(s) + df \otimes s.$$

Example 1.2. If $\mathscr{E} = \mathscr{O}_X$, the map $d : \mathscr{O}_X \to \Omega^1_{X/S}$ is a natural connection. More generally, every connection on $\mathscr{O}_X$ is of the form $f \mapsto df + f\omega$, for some $\omega \in \Omega^1_{X/S}$.

Indeed, because the difference of any two connections is linear, we observe:

Lemma 1.3. *The set of S-connections on $\mathscr{E}$ naturally form a $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_{X/S}$-pseudotorsor: i.e., if the set is non-empty, it is a torsor over $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_{X/S}$.*

Definition 1.4. We also recall that a **derivation** on $\mathscr{O}_X$ is an $\mathscr{O}_S$-linear map $\theta : \mathscr{O}_X \to \mathscr{O}_X$ to itself satisfying the Leibniz rule $\theta(fg) = \theta(f)g + f\theta(g)$.

Associated to θ is a unique $\mathscr{O}_X$-linear homomorphism $\hat{\theta}$ from $\Omega^1_{X/S}$ to $\mathscr{O}_X$, which gives θ upon precomposition with $d : \mathscr{O}_X \to \Omega^1_{X/S}$ (see [1, p. 386]).

If we apply these definitions after restriction to open subsets, we obtain natural sheaves of connections on $\mathscr{E}$ and derivations, which we denote by $\text{Conn}(\mathscr{E})$ and Der respectively. Note however that although Der is an $\mathscr{O}_X$-module (indeed, by the above it is isomorphic to $(\Omega^1_{X/S})^\vee$), $\text{Conn}(\mathscr{E})$ is just a sheaf of sets.

Observe that given $\nabla \in \text{Conn}(\mathscr{E})(U)$ and $\theta \in \text{Der}(U)$, we obtain a map $\nabla_\theta : \mathscr{E}|_U \to \mathscr{E}_U$ defined by $\nabla_\theta = (\hat{\theta} \otimes 1) \circ \nabla$.

Example 1.5. Again in the case $\mathscr{E} = \mathscr{O}_X$, with $X = \mathbb{P}^1_S$, let t be a coordinate for the line, and take $\theta = \frac{d}{dt}$. A differential form ω may be written as $g(t)dt$, so a connection ∇ is of the form $\nabla(f(t)) = df(t) + f(t)g(t)dt$. Then $\nabla_\theta(f(t)) = \frac{df(t)}{dt} + f(t)g(t)$; i.e., it is a 1st-order linear differential operator.

Definition 1.6. Given a connection ∇ on $\mathscr{E}$, we denote by $\mathscr{E}^{\nabla}$ the sheaf of sections on $\mathscr{E}$ which are in the kernel of ∇; these sections are often called **horizontal**, or **flat**.

We remark that connections, and in particular their horizontal sections, are very different sorts of objects depending on whether or not S is of positive characteristic.

Example 1.7. Consider the connection ∇ on $\mathscr{O}_{\mathbb{P}^1_S}$ given by $f(t) \mapsto df(t) - f(t)dt$. This has no horizontal sections algebraically, but formally locally, in characteristic 0, e^t is horizontal. However, in characteristic p there are no solutions even formally locally.

In characteristic 0, $\mathscr{E}^{\nabla}$ is very small; a finitely-generated $\mathscr{O}_S$-module. But in positive characteristic, one sees immediately that if $s \in \mathscr{E}^{\nabla}$, then $fs \in \mathscr{E}^{\nabla}$ for any $f \in \mathscr{O}_{X^{(p)}}$. Thus, although $\mathscr{E}^{\nabla}$ is not an $\mathscr{O}_X$-module, it is an $\mathscr{O}_{X^{(p)}}$ module. More generally, although ∇ is not $\mathscr{O}_X$-linear, it is $\mathscr{O}_{X^{(p)}}$-linear. Since $\mathscr{O}_X$ is a finite $\mathscr{O}_{X^{(p)}}$ module, one can sometimes treat ∇ as a linear object in positive characteristic, and one finds, for instance, that contrary to the case of characteristic 0, it is always sufficient to check for horizontal sections formally locally.

1.2. **Operations on sheaves with connection.** We observe that the operations of tensor product and homomorphism extend naturally to coherent sheaves with connection.

Given connections ∇_i on coherent sheaves $\mathscr{E}_i$ for $i = 1, \dots, m$, one can define a connection on $\mathscr{E}_1 \otimes \cdots \otimes \mathscr{E}_m$ by the formula $\nabla_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \nabla_2 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \nabla_m$.

For coherent sheaves $\mathscr{E}, \mathscr{F}$ with connections ∇, ∇', we have a canonical induced connection on $\mathcal{H}om(\mathscr{E}, \mathscr{F})$, given explicitly by $\varphi \mapsto \nabla' \circ \varphi - (\varphi \otimes 1) \circ \nabla$, and therefore its horizontal sections are precisely homomorphisms from $\mathscr{E}$ to $\mathscr{F}$ which commute with ∇, ∇'.

1.3. **Curvature.** We now define the notion of curvature of a connection. We first observe that given a connection ∇ on $\mathscr{E}$, for any i we obtain a map $\nabla_i : \Omega^i_{X/S} \otimes \mathscr{E} \to \Omega^{i+1}_{X/S} \otimes \mathscr{E}$ given by

$$\nabla_i(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \nabla(s).$$

Definition 1.8. The **curvature** K of ∇ is the map

$$\nabla_1 \circ \nabla : \mathscr{E} \to \Omega^2_{X/S} \otimes \mathscr{E}.$$

Even though neither ∇ nor ∇_1 is $\mathscr{O}_X$-linear, one can check that their composition is in fact $\mathscr{O}_X$-linear, so that $K(\nabla)$ may be thought of as an element of $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^2_{X/S}$.

Definition 1.9. We say that a connection ∇ is **integrable** if $K(\nabla) = 0$.

Example 1.10. If $\mathscr{E} = \mathscr{O}_X$, and ∇ is of the form $d + \omega$ for some $\omega \in \Omega^1_{X/S}$, then ∇ is integrable if and only if $d\omega = 0$; indeed, one can check from the definitions that $K(\nabla) = d\omega$.

The motivation for the terminology is that if $S = \operatorname{Spec} \mathbb{C}$, we have that ∇ is integrable if and only if there exist a full set of horizontal sections analytically locally.

We remark that if we consider ∇ as a map $\mathrm{Der} \to \mathcal{E}nd_{\mathscr{O}_S}(\mathscr{E})$ by $\theta \mapsto \nabla_\theta$, the curvature of ∇ measures the failure of ∇ to commute with the Lie bracket operations on Der and $\mathcal{E}nd_{\mathscr{O}_S}(\mathscr{E})$

1.4. **p-curvature.** In this section, we define the p-curvature of an integrable connection, which will function as a measure of the number of horizontal sections of ∇ in the case of positive characteristic; we will make this more precise shortly.

Definition 1.11. If $\mathscr{F}, \mathcal{G}$ are $\mathscr{O}_X$-modules, we also define a map $\varphi : \mathscr{F} \to \mathcal{G}$ to be **p-linear** if it is additive, and satisfies $\varphi(fs) = f^p \varphi(s)$ for $s \in \mathscr{F}(U)$, $f \in \mathscr{O}_X(U)$, and all open $U \subset X$.

Now, for any $\mathscr{O}_X$-module $\mathscr{F}$, it is easy to see that the natural map $\pi_{\mathscr{F}} : \mathscr{F} \to F_X^* \mathscr{F}$ gives a "universal p-linear map", which is to say, any p-linear map $\mathscr{F} \to \mathcal{G}$ factors through $\pi_{\mathscr{F}}$ to give a unique $\mathscr{O}_X$-linear map $F_X^* \mathscr{F} \to \mathcal{G}$.

If ∇ is an integrable connection on $\mathscr{E}$, then there is a notion of p-curvature associated to ∇ defined as follows: since we are in characteristic p, for any derivation θ, it is easy to check that θ^p is another derivation, and we define:

Definition 1.12. The **p-curvature** of ∇ is the map from $\mathrm{Der}(\mathscr{O}_X(U))$ to $\mathcal{E}nd_{\mathscr{O}_S}(\mathscr{E})$ given by

$$\psi_\nabla(\theta) = (\nabla_\theta)^p - \nabla_{(\theta^p)}.$$

Thus, just as curvature measures failure of the connection to commute with Lie bracket, p-curvature measures the failure of the connection to commute with pth powers.

We extend this to a sheaf morphism $\psi_\nabla : \mathrm{Der}(\mathscr{O}_X) \to \mathcal{E}nd_{\mathscr{O}_S}(\mathscr{E})$, and it turns out that it actually takes values in $\mathcal{E}nd_{\mathscr{O}_X}(\mathscr{E})$, and moreover is p-linear on $\mathrm{Der}(\mathscr{O}_X)$ (that is, $\psi_\nabla(f\theta) = f^p \psi_\nabla(\theta)$) (see [2, 5.0.5, 5.2.0]). The first result is easy to check, while the second requires an elementary but rather involved and unenlightening argument of Deligne.

Also, $\psi_\nabla(\theta)$ commutes with $\nabla_{\theta'}$ for any θ' [2, 5.2.3].

Finally, if we pull back a coherent sheaf $\mathscr{E}$ on $X^{(p)}$ under F, the sections of $F^*\mathscr{E}$ on $U \subset X$ will be described as $f \otimes s$, where $f \in \mathscr{O}_X(U), s \in \mathscr{E}(U)$, and we have $(F^* f) \otimes s = 1 \otimes fs$. We then see that we can have a **canonical connection** ∇^{can} on $F^*\mathscr{E}$ defined by $f \otimes s \mapsto s \otimes df$.

We already observed that the kernel of ∇ is naturally a $\mathscr{O}_{X^{(p)}}$-module. Moreover, if we have $\mathscr{F}$ on $X^{(p)}$, it is easy to see from the definition that the kernel of ∇^{can} on $F^*\mathscr{F}$ recovers $\mathscr{F}$, and that ∇^{can} is integrable. We also see that given a derivation θ, $\nabla^{\mathrm{can}}_\theta$ is given by $f \otimes s \mapsto (\theta f) \otimes s$, so that the p-curvature associated to ∇^{can} is visibly always 0.

This may seem suggestive, and indeed the Cartier theorem states that given a coherent sheaf $\mathscr{E}$ with a connection ∇ whose p-curvature vanishes, then $\mathscr{E}$ is the pullback of a coherent sheaf on $X^{(p)}$ under Frobenius, with ∇ being the corresponding canonical connection. One can even construct an appropriate categorical equivalence in this manner:

Theorem 1.13. *([2, 5.1]) Let X be a smooth S-scheme, with S having characteristic p, and let $F : X \to X^{(p)}$ be the relative Frobenius morphism. Then pullback under Frobenius (together with the associated canonical connection) and taking kernels of connections are mutually inverse functors, giving an equivalence of categories between the category of coherent sheaves on $X^{(p)}$ and the full subcategory* *of the category of coherent sheaves with integrable connection on X consisting of objects whose connection has p-curvature zero.*

Thus, the p-curvature of an integrable connection vanishes if and only if the sheaf $\mathscr{E}$ is generated by the subsheaf $\mathscr{E}^{\nabla}$ of horizontal sections. However, p-curvature is not only interesting for the case that it vanishes.

Definition 1.14. An integrable connection ∇ is said to be **nilpotent** of exponent $\leq n$ if given any derivations $\theta_1, \ldots, \theta_n$, we have $\psi_\nabla(\theta_1) \cdots \psi_\nabla(\theta_n) = 0$.

Katz showed the following:

Proposition 1.15. *An integrable connection ∇ is nilpotent of exponent $\leq n$ if and only if there exists a filtration $0 = F^n \subset F^{n-1} \subset \ldots F^1 \subset F^0 = \mathscr{E}$ such that ∇ induces a connection ∇^i on each graded piece F^i/F^{i+1}, and each ∇^i has vanishing p-curvature.*

In particular, if a connection is nilpotent, then $\nabla|_{F^{n-1}}$ has vanishing p-curvature, so $\mathscr{E}$ has horizontal sections.

Finally, we discuss p-curvature as a formal object. Since $\operatorname{Der}(\mathscr{O}_X) \cong (\Omega^1_{X/S})^\vee$, the p-linearity means we can consider ϕ_∇ as an $\mathscr{O}_X$-linear map $F_X^*(\Omega^1_{X/S})^\vee \to \mathcal{E}nd(\mathscr{E})$; compatibility of $\Omega^1_{X/S}$ with base change yields $\pi^*_{X/S}\Omega^1_{X/S} \cong \Omega^1_{X^{(p)}/S}$, so $F_X^*\Omega^1_{X/S} \cong F^*\Omega^1_{X^{(p)}/S}$, and we finally find we can consider p-curvature as giving a global section

$$\psi_\nabla \in \Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}/S}).$$

We claim that in fact, ψ_∇ lies in the kernel of the connection ∇^{ind} induced on $\mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}}$ by ∇ on $\mathscr{E}$ (inducing $\nabla^{\mathcal{E}nd}$ on $\mathcal{E}nd(\mathscr{E})$) and ∇^{can} on $F^*\Omega^1_{X^{(p)}}$. This follows formally from the fact that $\psi_\nabla(\theta)$ commutes with $\nabla_{\theta'}$ for all θ, θ'; if one thinks of ψ_∇ as a linear map $F^* \operatorname{Der}_{X^{(p)}} \to \mathcal{E}nd(\mathscr{E})$, one may use this commutativity to explicitly write down the actions of ∇^{can} and $\nabla^{\mathcal{E}nd}$, and see that they commute with ψ_∇. We therefore obtain the strengthened statement:

$$\psi_\nabla \in \Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}/S})^{\nabla^{\text{ind}}}, \tag{1.16}$$

where ∇^{ind} is the connection induced by ∇ and ∇^{can}.

We conclude with a brief discussion of the Cartier operator and a formula for p-curvature in rank 1.

We have:

Theorem 1.17. *There exists an isomorphism*

$$C : F_*\mathscr{H}^1(\Omega^*_{X/S}) \xrightarrow{\sim} \Omega^1_{X^{(p)}/S},$$

called the **Cartier operator**, *and with C^{-1} given by the following formula: if $f' \in \mathscr{O}_{X^{(p)}}$ is given by π^*f, where $\pi : X^{(p)} \to X$ is the base change of F_S, then*

$$C^{-1}(df') = [f^{p-1}df].$$

Example 1.18. Given the Cartier operator, we have the following formula for p-curvature of a connection, recalling that we could write any integrable connection on $\mathscr{O}_X$ as $d + \omega$ for some closed 1-form ω:

$$\psi_\nabla(\theta) = F^*(\pi^*\omega - C(\omega)).$$

2. The Grothendieck perspective

We now discuss Grothendieck's point of view on algebraic connections, which is more intrinsically geometric than the standard definition, particularly where the significance of integrability is concerned. We will also discuss a definition of p-curvature in this setting due to Mochizuki, and explain some simple applications.

2.1. Connections.

Let S be a scheme, and X smooth and finite type over S throughout.

We consider the diagonal map $\Delta : X \to X \times_S X$; $\Delta(X)$ is the closed subscheme corresponding to the ideal sheaf $\mathscr{I}$, which is generated by elements of the form $t \otimes 1 - 1 \otimes t$. We have:

Definition 2.1. The nth **infinitesimal neighborhood** of Δ is the closed subscheme $X^{(n)}$ corresponding to $\mathscr{I}^{n+1}$.

Thus, we have

$$X \hookrightarrow X^{(2)} \hookrightarrow X^{(3)} \hookrightarrow \dots X \times_S X.$$

We will denote by p_1, p_2 the projection maps $X \times_S X \to X$.

Given an object B over X (it could be a sheaf, or a scheme, or anything for which base change is defined), Grothendieck defines:

Definition 2.2. A **connection** on B is an isomorphism $(p_1^* B)|_{X^{(2)}} \xrightarrow{\sim} (p_2^* B)|_{X^{(2)}}$ which is the identity map when restricted to Δ.

We do not explain why this definition is equivalent to the standard one in the case that B is a sheaf. However, the relationship comes about largely because $p_{1*}(\mathscr{I}/\mathscr{I}^2)$, then is naturally isomorphic to $\Omega^1_{X/S}$, with dt corresponding to $t \otimes 1 - 1 \otimes t$.

We focus instead on discussing the geometry of this definition. As Ishai mentioned, it should be thought of as giving "first-order parallel transport". If we try to picture what this might mean algebraically, we ought to suppose that S is defined over some field k, and we let (T, x) be any pointed S-scheme. If we have a map $f : (T, x) \to (X, x_0)$ (one may imagine this is a closed immersion), we can also consider the constant map $f_{x_0} : (T, x) \to (X, x_0)$ sending all of T to x_0, it would make sense to define parallel transport along $f(T)$ from x_0 as an isomorphism $B|_{f(T,x)} \xrightarrow{\sim} B|_{f_0(T,x)}$ which gives the identity at x_0; that is, a trivialization the restriction of B to the image of T which is given as an isomorphism between this and the trivial bundle on T having fiber $B|_{x_0}$ everywhere.

In this case, first-order parallel transport could reasonably mean that we have this sort of parallel transport for any f with $T = \operatorname{Spec} k[\epsilon]/\epsilon^2$, which one ought to think of as a point together with a tangent vector. A map $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to X$ gives a point by composing with the natural map $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\epsilon]/\epsilon^2$, and a tangent vector at that point.

To see that we have such data from a connection, take some f and f_0, then we obtain a map $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\epsilon]/\epsilon^2 \hookrightarrow X \times_S X$. Because the image of $\operatorname{Spec} k$ lies in Δ, and $\epsilon^2 = 0$, it is easy to check that this map factors through $X^{(2)}$, so we can pull back the connection to obtain an isomorphism of $p_1^* B|_{\operatorname{Spec} k[\epsilon]/\epsilon^2} \xrightarrow{\sim} p_2^* B|_{\operatorname{Spec} k[\epsilon]/\epsilon^2}$ which restricts to the identity on $\operatorname{Spec} k$. But because of the construction, this is precisely an identification of $B|_{f(\operatorname{Spec} k[\epsilon]/\epsilon^2)}$ with $B|_{f_0(\operatorname{Spec} k[\epsilon]/\epsilon^2)}$, which is what we wanted.

Thus, we can think of a connection as giving us parallel transport along the shortest possible paths in X – tangent vectors.

2.2. **Integrable connections.** In this Grothendieckian setting, integrability is expressed by a cocycle condition:

Definition 2.3. Consider the triple product $X \times_S X \times_S X$, and denote by $X_3^{(2)}$ the first infinitesimal neighborhood of the diagonal in the product. We have projection maps $p_{i,j}$ to $X \times_S X$. Then given a connection, we obtain isomorphisms $\epsilon_{i,j} : p_i^* B|_{X_3^{(2)}} \overset{\sim}{\to} p_j^* B|_{X_3^{(2)}}$ by pulling back the connection under $p_{i,j}$ and restricting. The connection is **integrable** if

$$\epsilon_{1,3} = \epsilon_{2,3} \circ \epsilon_{1,2}.$$

In characteristic 0, one shows:

Theorem 2.4. *If a connection is integrable, it can be lifted to nth-order neighborhoods for any n, that is, it gives (a compatible system of) isomorphisms $p_1^* B|_{X^{(n)}} \overset{\sim}{\to} p_2^* B|_{X^{(n)}}$.*

By the same argument as above, these give nth-order parallel transport data, which is to say, given maps $\operatorname{Spec} k[\epsilon]/\epsilon^{n+1} \to X$, we obtain trivializations of $B|_{\operatorname{Spec} k[\epsilon]/\epsilon^{n+1}}$ as before.

Because these are compatible, we can also pass to limits, obtaining trivializations of B along formal paths $\operatorname{Spec} k[[t]]$. Indeed, the same argument works to produce trivializations of B along entire formal neighborhoods of any point in X, which can then be used to produce a family (formally locally) of horizontal sections of full rank, as should exist for an integrable connection in characteristic 0.

Unfortunately, one cannot expect such behavior in characteristic p. However, it turns out that one can still perform such liftings if one slightly modifies the construction of $X^{(n)}$ using PD-structures. The main object we use is the PD-scheme $X \times_S^{\mathrm{PD}} X$, which is by definition the **PD envelope** of the map of standard schemes $\Delta : X \hookrightarrow X \times_S X$.

Formally, a PD-scheme is a triple $(Y, \mathscr{J}, \gamma)$ where Y is a scheme, $\mathscr{J}$ an ideal sheaf, and γ is a compatible collection of maps $\gamma_n : \mathscr{J} \to \mathscr{O}_Y$ satisfying the conditions from Ishai's talk. One should think of $\mathscr{J}, \gamma$ as additional information allowing one to consider $\mathscr{O}_Y$ to be containing elements "$t^n/n!$" for any $t \in \mathscr{J}$. Note that this doesn't make sense purely algebraically, because if we write $t^{[n]}$ for this element, if $p|n$ the only relations we obtain are along the lines of $n! \cdot t^{[n]} = t^n$, which forces both sides to be 0. This is reason for the necessity of the additional data γ to obtain a meaningful theory.

Now, $X \times_S^{\mathrm{PD}} X$ is the PD-scheme obtained from $X \times_S X$ and the ideal $\mathscr{I}$ in order to include $\mathscr{I}^{[n]} := (t^{[n]} : t \in \mathscr{I})$. We can thus define

Definition 2.5. The nth infinitesimal PD-neighborhood of Δ is the closed PD-subscheme $X^{[n]}$ of $X \times_S^{\mathrm{PD}} X$ cut out by $\mathscr{I}^{[n+1]}$.

Given this construction, one can prove the following (rather non-trivial) theorem:

Theorem 2.6. *A connection on an $\mathscr{O}_X$-module can be lifted compatibly to $X^{[n]}$ for all n if and only if it is integrable.*

The proof is rather involved, and goes through a correspondence with modules over the ring of PD-differential operators. This argument relies on the fact that the connection is on an $\mathscr{O}_X$-module. Since Grothendieck's definition of integrable connection and the statement on liftability rely only on understanding X/S, and don't involve any properties of B beyond the existence of base change, it is natural to ask:

Question 2.7. Is there an "intrinsic" argument for the above theorem not relying on any special properties of B?

2.3. **Mochizuki's p-curvature.** We outline a construction of p-curvature in the Grothendieck setting due to Mochizuki. We first give an interpretation of $F^*\Omega^1 X^{(p)}/S$ in terms of PD-neighborhoods:

Proposition 2.8. *Let $(X \times_S^{\mathrm{PD}} X, \mathscr{J}, \gamma)$ be the* PD*-envelope of the diagonal, and $\mathscr{I}$ the ideal of the standard diagonal Δ, considered inside $\mathscr{O}_{X\times_S^{\mathrm{PD}} X}$. Then there is a natural isomorphism*

$$F^*\Omega^1 X^{(p)}/S \to \mathscr{J}/(\mathscr{J}^{[p+1]}, \mathscr{I}).$$

The picture is as follows:

$$\begin{array}{ccccc} X \times_S^{\mathrm{PD}} X & \hookleftarrow & V(\mathscr{I}) & \hookleftarrow & V(\mathscr{I}, \mathscr{J}^{[p+1]}) \\ \downarrow & & \downarrow & & \\ X \times_S X & \hookleftarrow & X & & \end{array}$$

Note that the difference between $\mathscr{I}$ and $\mathscr{J}$ is that $\mathscr{J}$ contains all its PD-powers, whereas $\mathscr{I}$ only contains standard powers. For the proof of this proposition, see [3, Prop. 1.4]. The proposition implies that the structure sheaf of $V(\mathscr{I}, \mathscr{J}^{[p+1]})$ is isomorphism of $\mathscr{O}_X \oplus F^*\Omega^1_{X^{(p)}/S}$, with multiplication given by a square-zero structure on the 2nd summand.

Now we assume we are given an integrable connection. We know from the theorem that we can lift it to $X^{[p+1]}$, which is $V(\mathscr{J}^{[p+1]})$. If we restrict further to $V(\mathscr{I}, \mathscr{J}^{[p+1]})$, we obtain an isomorphism of p_1^*B with p_2^*B which restricts to the identity on $V(\mathscr{J})$; we also have the tautological isomorphism obtained by restricting to $\mathscr{I}$, and composing the first with the inverse of the second, we obtain an automorphism of p_1^*B on $V(\mathscr{I}, \mathscr{J}^{[p+1]})$, which restricts to the identity modulo $\mathscr{J}$. Because of the square-zero structure on the ideal sheaf of $V(\mathscr{I}, \mathscr{J}^{[p+1]})$, this gives a section of $\mathrm{InfAut}(B) \otimes \Omega^1_{X^{(p)}/S}$, where $\mathrm{InfAut}(B)$ is the sheaf of infinitesimal automorphisms of B.

Definition 2.9. The ***p*-curvature** of an integrable connection on B is defined to be the above-constructed section of $\mathrm{InfAut}(B) \otimes \Omega^1_{X^{(p)}/S}$.

In the case that B is a locally free $\mathscr{O}_X$-module, this is identified with $\mathcal{E}nd(B)$, and we have the following:

Proposition 2.10. *If B is an $\mathscr{O}_X$-module, then Mochizuki's definition of p-curvature agrees with the usual one.*

See [3, Prop. 1.5] for the proof in the general setting.

We remark that this definition of p-curvature has much to recommend it, as one can see abstractly many properties of p-curvature that are non-trivial in the traditional setting. For instance, it is by construction p-linear, and is easily seen to commute with base change. It also commutes well with operations on bundles, so it is easy to see that the p-curvature of a tensor product connection is the sum of the p-curvatures of the original connections, that the p-curvature of a determinant connection is the trace of p-curvature of the original connection, that the p-curvature of the projectivization of a connection is the traceless part of the p-curvature of the original connection, and so forth.

We also remark that vanishing of the p-curvature allows lifting of the connection to pth-order neighborhoods in the standard, non-PD setting. However, it does not necessarily allow lifting to all orders, which requires a stronger condition, in some sense an infinite iteration of the property of having p-curvature zero.

References

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