FACTS ABOUT (INVERSE) CARTIER TRANSFORM.

Chow ring.

$$K^0(X) \xrightarrow{ch} CH^*(X)$$

Recall X sm proj $/k$, $\tilde{X}$ lift to W_2. Constructed functors , $X \xrightarrow{F} X'$

$$\mathrm{MIC}_{p-1}(X) \underset{C^{-1}_{\tilde{X}}}{\overset{C_{\tilde{X}}}{\rightleftarrows}} \mathrm{HIG}_{p-1}(X')$$

sheaves with int. connection of exp $\leq p-1$

Higgs sheaves of exp $\leq p-1$

$F: X \to X'$ relative Frob.

By Ogus-Vologodsky these are eq. of cats.

[OV, Cor 2.27]

Thm $(E, D) \in \mathrm{MIC}_{\leq \ell}(X/S)$ with $\ell + \dim X < p$.

$\exists$ isom in the derived cat.

$$F_{X*}(E \otimes \Omega^\bullet_X) \cong C_{\tilde{X}}(E) \otimes \Omega^\bullet_{X'}$$

Recovers DI when $(E, D) = (\mathcal{O}, d)$

Grothe. Sp

Recall $K_0(X) = \langle [E] \mid E \text{ coherent sheaf} \rangle / [E]=[F]+[G]$ for all $0 \to F \to E \to G \to 0$.

Prop $[C^{-1}_X((E,\theta))] = [F_X^* E] \in K_0(X)$.

Pf. Recall (E,θ) has exponent $\leq p-1 \Rightarrow$ can filter by sub Higgs sheaves

$F_1 \subset F_2 \subset \dots \subset (E,\theta)$ s.t. assoc. gr

have zero Higgs field. $[E] = [gr^\circ E] \in K_0(X)$, so

suffice to prove for sheaves w/ trivial Higgs field, which holds by construction.

§ Semi-stability

Assume (E) is torsion free. Pick $D \subset X$ ample

$$\mu(E) = \frac{c_1(E) \cdot D^{\dim X - 1}}{\operatorname{rank}(E)}.$$

[In char 0, only (?) semi-stable Higgs bundles are interesting.]

Recall **Def** (E,θ) is slope ss if $\forall (F,\theta) \subset (E,\theta)$

$$\mu(F) \leq \mu(E,\theta).$$

Have immediate:

Cor (E,θ) is slope ss iff $C^{-1}(E,\theta)$ is .

§ **HdR-flows**

let $C_1^{-1} := C^{-1}_{X} \circ \pi^* \circ \iota$ where
$$\begin{tikzcd} X \arrow[r] & X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

and $\iota : HIG_{p-1}(X) \to HIG_{p-1}(X)$

is $(E,\theta) \longmapsto (E,-\theta)$.

__Defn__ (HdR-flow) • $(E_0,\theta_0) \overset{C_1^{-1}}{\rightsquigarrow} (H_0,D_0) \overset{\text{for } Fil_0^\bullet}{\rightsquigarrow} (E_1,\theta_1) \rightsquigarrow \cdots \rightsquigarrow (E$

where $Fil_i^\bullet$ is Griffiths transverse filtration on (H_i, D_i)

• periodic if $\exists$ $\phi : (E_f, \theta_f) \cong (E_0, \theta_0)$

• pre-periodic if periodic after removing first few terms.

Rmk $\exists$ versions over $W_2(k), W_3(k), \ldots,$ and $W(k)$.

Thm $\mathcal{X}/W$ $\exists$ equivalence w

$$\left\{\begin{array}{l}\text{periodic HdR} \\ \text{flows on } \mathcal{X} \\ \text{of level } \le p-1, \text{ period}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}p\text{-torsion free} \\ \text{Fontaine-modules} \\ \text{w/ endo}^n \text{ by } W(\mathbb{F}_{p^f}) \\ \text{of Hodge-Tate weights} \\ \le p-1.\end{array}\right\}$$

means $H = F^0 \supset F^1 \supset \ldots \supset F^w \supset F^{w+1} = 0$.

crystalline repn

$$\pi_1^{\text{et}}(\mathcal{X}_K) \longrightarrow GL_r(\mathbb{Q}_{p^f})$$

rank one example

Example Lubin-Tate characters.

Thm (OV) If $(E,\theta) = R^i\pi_*^{Higgs}\mathcal{O}_Y$, $\quad Y \xrightarrow{\pi} \overset{\text{smooth}}{X}$, then a lift $\tilde{Y} \xrightarrow{\tilde{\pi}} \tilde{X} \,/\, W_2(k)$

induces $C^{-1}(E,\theta) \cong R^i\pi_*^{dR}\mathcal{O}_Y$.

Following was first proved by Simpson over $\mathbb{C}$.

<u>Thm</u> (Langer) (E, θ) rank $\leq p$ slope ss Higgs sheaf, with vanishing Chern classes.
Then E is locally free.

Thm X/k sm proj., X_2/W_2 lift.
preperiodic Higgs module is slope semistable with vanish. Chern classes.

Pf Suppose the sequence of Higgs bdle is

$$(E_0, \theta_0), (E_1, \theta_1), \ldots.$$

$$[E_i] = [(F^*)^i(E_0)] \in K_0(X)$$

and hence

$$c_\ell(E_i) = p^{\ell i} c_\ell(E_0)$$

preperiodic $\Rightarrow$ $c_\ell(E_0) = 0$ in $CH(X) \otimes \mathbb{Q}$.

If $(F, \theta) \subset (E, \theta)$ with $\mu(F, \theta) > 0$
then again preperiodic $\rightarrow$ (E_i, θ_i) (some i)
has sub-Higgs sheaves of arb. large μ, ✗.

Rmk converse holds under some assumptions. More precisely
$(E, \theta) \in HIG_{\leq p-1}$. Then $rk(E) \leq p$, $c_i(E) = 0$ $\forall i > 0$ ss $\Rightarrow$ pre-periodic.

Cor. if $\pi : Y \xrightarrow{\text{sm proper}} X$ lifts to $\tilde{Y} \to \tilde{X}$ then

$R^i \pi_*^{\log} \mathcal{O}$ is semi-stable.

Question.

- any fam of generically ord. AV's has associated Higgs bundle semi-stable?
- Not true w/o ordinary assumption

Rmk:

- easy to find pre-periodic (even nilpotent) Higgs bundles which are not periodic.
- example of families $Y \to X$ with ss Higgs bundle w/o lifting to W_2?