Global gole: $\rho : \pi_1(\Sigma_{g,n}) \longrightarrow GL_r(\mathbb{C})$,

ρ is MCG-finite $\left(\text{that is, } |MCG_{g,n} \cdot [\rho]| < \infty,\ [\rho] \in M_B^r(\Sigma_{g,n})\right)$

If $r < \sqrt{g+1}$, then $|\operatorname{Im} \rho| < \infty$

Set-up:

$$\begin{tikzcd} \mathcal{C}^\circ \arrow[r, "j"] \arrow[dr, "\pi^\circ"'] & \mathcal{C} \arrow[d, "\pi"] & \mathcal{D}_1 \arrow[l] \cdots & \mathcal{D}_n \arrow[ll, bend right] \\ & \mathcal{M} \arrow[ur, "s_1 \ \cdots"] \arrow[urr, "s_n"'] & & \end{tikzcd}$$

$\mathcal{C}/\mathcal{M}$ - versal family of (g,n)-curves

versal: $c : \mathcal{M} \longrightarrow \mathcal{M}_{g,n}$ is dominant étale

$s_1, \ldots, s_n : \mathcal{M} \longrightarrow \mathcal{C}$ non-intersecting sections ("punctures")

$\mathcal{D}_i := s_i(\mathcal{M})$, $\mathcal{D} := \bigcup \mathcal{D}_i$

$\mathcal{C}^\circ := \mathcal{C} \setminus \mathcal{D}$ family of quasi-projective curves

$\mathbb{V} \longrightarrow \mathcal{C}^\circ$ unitary semisimple local system, $\operatorname{rk} \mathbb{V} = r$

Theorem (1.7.1): $R^1\pi^\circ_* \mathbb{V}$ has no sub-local systems of law rank:

$$\mathbb{L} \subset R^1\pi^\circ_* \mathbb{V},\ \mathbb{L} \neq 0 \Rightarrow \operatorname{rk} \mathbb{L} \geq 2g - 2r$$

Remarks:

1) Corollary: if $r<g$ then $H^0(\mathcal{M}, R^1\pi_*^\circ \mathbb{V})=0$
(otherwise there is a rk 1 subsystem generated by an invariant vector, but $\mathrm{rk}\,\mathbb{L} \geq 2$)

2) What if $\mathbb{V}=\underline{\mathbb{C}}_{\mathcal{C}^\circ}$?

$$MCG_{g,n} \xrightarrow{\sigma} Sp_{2g}(\mathbb{Z}) \circlearrowright H^1(\Sigma_g, \mathbb{C})$$

$$\sigma(T_\alpha)\cdot[\beta] = [\beta] + \langle \alpha, \beta \rangle [\alpha] \quad (\text{"Picard-Lefschetz"})$$

T_α → Dehn twist around α

$\Rightarrow \sigma$ is surjective $\Rightarrow H^0(\sigma)=0$

3) Assume $\mathbb{V}$ is fiberwise irreducible. Apply to
$\mathrm{ad}\,\mathbb{V} = \mathrm{End}(\mathbb{V})/\mathbb{C}\cdot \mathrm{id}_{\mathbb{V}}$.
$\mathrm{rk}\,\mathrm{ad}\,\mathbb{V} = r^2-1 < g$ if $r < \sqrt{g+1}$

- By 1): $H^0(\mathcal{M}, R^1\pi_*^\circ \mathrm{ad}\,\mathbb{V})=0$
- By Schur lemma: $\pi_*^\circ \mathrm{ad}\,\mathbb{V}=0 \Rightarrow H^1(\mathcal{M}, \pi_*^\circ \mathrm{ad}\,\mathbb{V})=0$
- By Leray sp. seq.: $H^1(\mathcal{C}^\circ, \mathrm{ad}\,\mathbb{V})=0$
(rigidity result!!!)

1. VHS on $R^1\pi_*^\circ \mathbb{V}$

If $\mathbb{V}$ is constant, one may study $R^\bullet \pi_* \mathbb{V}$ using Hodge theory. What if $\mathbb{V}$ is locally constant?

Assume $\mathbb{V}$ is real $(\exists\, \mathbb{V}_{\mathbb{R}} \in \mathrm{Loc}_{\mathbb{R}}(\mathcal{C}^\circ) : \mathbb{V} = \mathbb{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$

[N.B.: $\mathbb{V}$ is unitary $\Rightarrow \mathbb{V} \oplus \mathbb{V}^\vee$ is real]

$\mathcal{H}^{i,j}_{\mathbb{V}} = \mathbb{V} \cap \mathcal{H}^{i,j}_{(\mathbb{V}\oplus\mathbb{V}^\vee)}$

Fact: $R^1 \pi^\circ_* \mathbb{V}$ underlies an admissible polarisable $\mathbb{R}$-MVHS.

Weight filtration:

$$\begin{array}{ccc} R^1\pi_* j_* \mathbb{V} & \hookrightarrow & R^1 \pi^\circ_* \mathbb{V} \\ \| & & \| \\ W^1 & & W^2 \end{array}$$

Hodge filtration:

Let $(\mathcal{E}, \nabla)$ be Deligne canonical extension of $\mathbb{V} \otimes \mathcal{O}_{\mathcal{C}^\circ}$ on $(\mathcal{C}, \mathcal{D})$.

De Rham complex:

$$\left[0 \to j_* \mathbb{V} \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{\mathcal{C}}(\log \mathcal{D})\right] = DR^\bullet(\mathcal{E})$$

Fact (Hodge-to-de Rham degeneration):

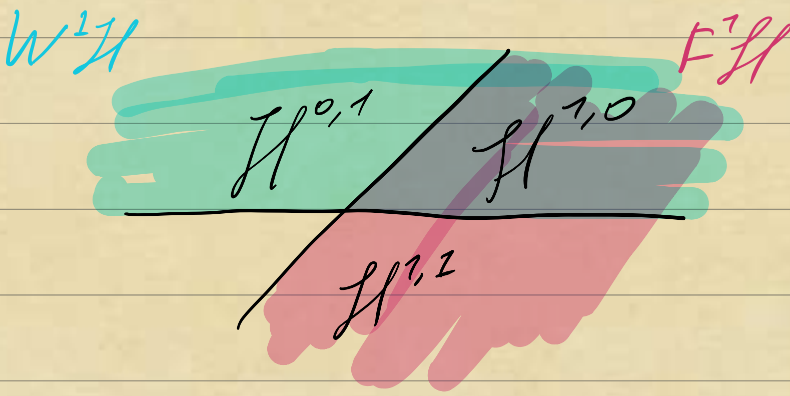
$$R^1\pi^\circ_* \mathbb{V} = \mathbb{R}^1\pi_*(DR^\bullet(\mathcal{E}))$$

filtration bête $\rightsquigarrow$ Hodge filtration

More explicitly: $\mathcal{H} := R^1\pi^\circ_* \mathbb{V} \otimes \mathcal{O}_M$

$$F^1\mathcal{H} = \mathrm{im}\left(\pi_*(\mathcal{E}\otimes\Omega^1_C(\log \mathcal{D})) \longrightarrow R^1\pi^\circ_*\mathbb{V}\right)$$

$$\mathcal{H}/F^1\mathcal{H} \simeq R^1\pi_*\mathcal{E}$$



2. Period map:

Let $\mathcal{B} \subset M$ be a ball.

Locally we have a period map

$$P : \mathcal{B} \longrightarrow Gr(f^1; h)$$
$$b \longmapsto F^1\mathcal{H} \subseteq \mathcal{H}$$

$\mathrm{rk}\, F^1\mathcal{H} = f^1$, $\mathrm{rk}\,\mathcal{H} = h$

$$dP : T\mathcal{B} \longrightarrow T\,Gr(f^1; h) = \mathrm{Hom}(F^1\mathcal{H}, \mathcal{H}/F^1\mathcal{H})$$

Fix a point $b \in \mathcal{B}$. Let $C := \mathcal{C}_b$

$$(C^\circ = \mathcal{C}^\circ_b,\ D = \mathcal{D}_b,\ E = \mathcal{E}|_C)$$

Our aim is to understand

$$dP_b : T_b\mathcal{B} \longrightarrow (F^1\mathcal{H})^\vee \otimes (\mathcal{H}/F^1\mathcal{H})$$

Dually: $F^1\mathcal{H}\otimes(\mathcal{H}/F^1\mathcal{H})^\vee \xrightarrow{dP^\vee} \Omega^1_b\mathcal{B}$

$c : \mathcal{M} \longrightarrow \mathcal{M}_{g,n}$ is dominant étale $\Rightarrow$

$c^* : \Omega^1_b\mathcal{B} \xrightarrow{\sim} H^0(C, \omega_C^{\otimes 2}(D)) = \Omega^1_{[C]}\mathcal{M}_{g,n}$

$$\begin{array}{ccccc}
F^1\mathcal{H}_b & \otimes & (\mathcal{H}/F^1\mathcal{H})_b^\vee & \longrightarrow & \Omega^1_b\mathcal{B} \\
\| & & \| & & \\
H^0(C, E\otimes\Omega^1_C(\log D)) & \otimes & H^1(C,E)^\vee & \xrightarrow{dP^\vee} & H^0(C, \omega_C^{\otimes 2}(D)) \\
\| & & \| \text{ Serre duality} & & \| \\
H^0(C, E\otimes\Omega^1_C(\log D)) & \otimes & H^0(C, E^\vee\otimes\Omega^1_C(\log D)) & \longrightarrow & H^0(C,\omega_C^{\otimes 2}(D))
\end{array}$$

Formal check: $dP^\vee$ coincides with the composition

$$\begin{tikzcd}
H^0(C,E\otimes\omega_C(\log D))\otimes H^0(C,E^\vee\otimes\omega_C(\log D)) \arrow[r, "\otimes"] \arrow[dr] & H^0(C,E\otimes E^\vee\otimes\omega_C^{\otimes 2}(\log D)) \arrow[d, "\mathrm{tr}\otimes \mathrm{id}"] \\
& H^0(C,\omega_C^{\otimes 2}(\log D))
\end{tikzcd}$$

(A well-known particular case:

$E=\mathcal{O}_C$, $n=0$:

$$\begin{array}{ccc}
H^{1,0}(C)\otimes H^{0,1}(C)^\vee & \longrightarrow & H^0(C,K_C^{\otimes 2}) \\
\| & & \| \\
H^0(C,\omega_C)\otimes H^0(C,\omega_C) & \xrightarrow{\otimes} & H^0(C,K_C^{\otimes 2})
\end{array}$$

)

The period map differential is the same as the map defined by the <u>Gauß-Manin connection</u>:

$$\begin{tikzcd} F^1\mathcal{H} \arrow[r, hook] \arrow[rrr, bend right, "\overline{\nabla}"] & \mathcal{H} \arrow[r, "\nabla_{GM}"] & \mathcal{H}\otimes\Omega^1_M \arrow[r] & (\mathcal{H}/F^1\mathcal{H})\otimes\Omega^1_M \end{tikzcd}$$

(By Griffiths transversality this is a linear map!)

<u>Polarisation</u>: One can think about this map as a pairing

$$B_E: H^0(C, E\otimes\omega_C(D))\otimes H^0(C, E^\vee\otimes\omega_C(D)) \longrightarrow H^0(C, \omega_C^{\otimes 2}(D))$$

More precisely:

$$\overline{\nabla}: v \longmapsto B_E(v,-).$$

<u>Idea of the proof</u> of Thm 1.7.1:

$\forall v$ we get a map $\overline{\nabla}(v): E \to (\mathcal{H}/F^1\mathcal{H})\otimes\Omega^1_b M$

We will show:

- If rk $\mathrm{Im}(\overline{\nabla}(v))$ on H^0 is big, then rk E is big
- If $\exists\, \mathbb{U}\subset E$ of small rank, then $\exists v$:

  $\mathrm{Im}(\overline{\nabla}(v))$ is of

3. Bound on $\operatorname{rk}\operatorname{Ker}(\overline{\nabla}(v))$

small rank

Prop. 6.3.6 from 2202.00039 ("non-GGG lemma"):

C - smooth proper curve of genus g
$D \subset C$ - effective reduced divisor
$E \longrightarrow C \setminus D$ vector bundle
$E_\star$ parabolic structure, semistable
$\widehat{E}_\star$ - coparabolically stable
$\mathcal{U} \subset \widehat{E}_0$ subbundle
$c := \operatorname{rk}\widehat{E}_0 - \operatorname{rk}\mathcal{U}$
$\delta := h^0(C, \widehat{E}_0) - h^0(C, \mathcal{U})$
If $\mu_\star(E_\star) > 2g-2+n$, then $\operatorname{rk} E > gc - \delta$
If $\mu_\star(E_\star) = 2g-2+n$, then $\operatorname{rk} E \geq gc - \delta$.

Proposition (4.2.3.):

$E_\star$ is a parabolic semistable on (C,D)
$\operatorname{degpar}(E_\star)$ $\quad 0 \neq v \in H^0(C, E \otimes \omega_C(D))$

Suppose $f_v := B_E(v, -)$, $f_v : E^\vee \otimes \omega_C(D) \longrightarrow \omega_C^{\otimes 2}(D)$

$H^0(C, f_v) : H^0(C, E^\vee \otimes \omega_C(D)) \longrightarrow H^0(C, \omega_C^{\otimes 2}(D))$
has rank r. Then $\operatorname{rk}(E) \geq g - r$.

Proof: Let $n = \deg D$

$\mathcal{U} = \ker f_v = v^{\perp}$ has corank 1, so
$c = 1$

Since $0 \to H^0(C, \mathcal{U}) \to H^0(C, E^\vee \otimes \omega_C(D)) \to \mathrm{Im}\, H^0(f_v) \to 0$
$\delta = r$

$\widehat{(E_*)^\vee \otimes \omega_C(D)}$ is Serre-dual to E_*

Since $\mu_*(E_*) = 0$, $\mu_*(\widehat{(E_*)^\vee \otimes \omega_C(D)}) = 2g-2+n$

But one can check:

$\widehat{(E_*)^\vee \otimes \omega_C(D)}_0 = E^\vee \otimes \omega_C(D)$ and

By non-GGG lemma $\mathrm{rk}\, E \geq g - r$

4. VHS on sub-local systems:

Let $\mathbb{L} \subset R^1\pi_*^\circ \mathbb{V}$.

Define $\overline{\mathbb{L}} := \begin{cases} \mathbb{L}, & \text{if } \mathbb{L} \text{ has } \mathbb{R}\text{-structure} \\ \mathbb{L} \oplus \overline{\mathbb{L}}, & \text{otherwise.} \end{cases}$

Clearly, $\widetilde{R^1\pi_* \mathbb{V}} = R^1\pi_* \widetilde{\mathbb{V}} =: \mathbb{W}$

$\mathbb{L} \subset \mathbb{W}_{\mathbb{C}}$. There exists i, s.t.
$\mathbb{L} \to gr^i_F \mathbb{W}_{\mathbb{C}}$ is non-zero

By Deligne semi-simplicity:

$$gr^i_W \mathbb{W}_{\mathbb{C}} = \bigoplus_j \mathbb{V}_j \otimes W_j \text{ , where}$$

$\mathbb{V}_j$ are irr. PVHS on M, W_j are Hodge str.-s

Enough to assume that $\mathbb{L}$ is irreducible (if $\mathbb{L}' \subseteq \mathbb{L}$ is irr., rank bound on $\mathbb{L}' \Rightarrow$ rank bound on $\mathbb{L}$)

Therefore, $\exists j$: $\mathbb{L} \simeq \mathbb{V}_j$. Thus $\mathbb{L}$ underlies a pure VHS!

The morphism $\mathbb{L} \hookrightarrow \mathbb{W}$ is not necessary a morphism of VHS, but:

$$Q := Hom_{\mathbb{R}\text{-VHS}}(\bar{\mathbb{L}}, \mathbb{W}) = H^0(M, \underbrace{\overbrace{\bar{\mathbb{L}}^\vee}^{\text{pure VHS}} \otimes \overbrace{\mathbb{W}}^{\text{admissible VMHS}}}_{\text{admissible VMHS}})$$

By fixed part theorem for admissible VMHS the space Q is endowed with $\mathbb{R}$-MHS and the natural map

$$\iota : Q \otimes \bar{\mathbb{L}} \longrightarrow \mathbb{W}$$

is a non-zero morphism of $\mathbb{R}$-VMHS. This is not true in general! However, it is true in this particular case, see $\otimes$ in

Let $Q \otimes \mathcal{L} := (Q \otimes \bar{\mathbb{L}}) \otimes \mathcal{O}_M$.

If $v \in F^1(Q\otimes L)$ we have

the end of these notes

$$\begin{tikzcd}
(Q\otimes L)/F^1(Q\otimes L) \arrow[r, "f_v = \overline{\nabla}_L(v)"] \arrow[d, "z \bmod F^1"] & T_b^* B \arrow[d, equal] \\
H/F^1H \arrow[r, "f_{z(v)} = \overline{\nabla}_H(z(v))"] & T_b^* B
\end{tikzcd}$$

Since z is a morphism of VHS

5. Rank bound for the differential of period map on $\mathbb{L}$

(5.1.1)

Lemma: Up to replacing $\mathbb{V} \rightsquigarrow \overline{\mathbb{V}}$, $\mathbb{L} \rightsquigarrow \overline{\mathbb{L}}$

there exists $v \in F^{\#}H_m \cap \mathbb{L}$, s.t. $\operatorname{rk} \overline{\nabla}_m(v) \leq \frac{\operatorname{rk} \mathbb{L}}{2}$.

Proof: As we explained, $\mathbb{L}$ underlies a pure VHS and there is a map

$z : Q \otimes \mathbb{L} \longrightarrow R^1 \pi_*^\circ \mathbb{V} = \mathbb{W}$

We have a diagram

$$F^1H \xrightarrow{\overline{\nabla}_W} H/F^1H \otimes \Omega^1_{\mu}$$

$$F'(Q\otimes L) \xrightarrow{\overline{\nabla}_{Q\otimes L}} (Q\otimes L)/F'(Q\otimes L) \otimes \Omega'_M$$

By duality:

$$\begin{tikzcd}
(\mathbb{W}/F'\mathbb{W})^\vee \arrow[r, "\overline{\nabla}_{\mathbb{W}}(\ell(v))"] \arrow[d] & \Omega'_M \\
\left((Q\otimes L)/F'(Q\otimes L)\right)^\vee \arrow[r, "\overline{\nabla}_{Q\otimes L}(v)"] & \Omega'_M \arrow[u]
\end{tikzcd}$$

$rk\, \overline{\nabla}_{\mathbb{W}}(\ell(v)) \le rk\, \overline{\nabla}_{Q\otimes L}(v)$; since Q is constant $rk\, \overline{\nabla}_{Q\otimes L}(v) = rk\, \overline{\nabla}_{\mathbb{L}}(v)$

$\mathbb{W} = \mathbb{W}^{1,0} \oplus \mathbb{W}^{0,1} \oplus \mathbb{W}^{1,1}$

We have a morphism of MVHS

$$Q\otimes \tilde{\mathbb{L}} \longrightarrow \mathbb{W}.$$

$\mathbb{L}$ is pure, so there are two possibilites (maybe up to twist)

1) $\mathbb{L} = \mathbb{L}^{0,0}$ and $Q = Q^{1,0} \oplus Q^{0,1} \oplus Q^{1,1}$

But then $\mathbb{L}$ does not vary at all!
(and Q is a constant variation

$\forall v \in F'(Q\otimes L)$ the map

$$\overline{\nabla}(v) : \left(Q\otimes L/\ \ \right)^\vee \longrightarrow H^0(C, End\ldots(D))$$

$\nabla_{\mathbb{L}}(v)\ \left(Q\otimes\mathbb{L}/F'(Q\otimes\mathbb{L})\right) \quad \to H(C, L\otimes\omega_C(D))$

vanishes. Therefore, $rk\,\overline{\nabla}_{\mathbb{W}}(z(v)) \leq rk\,\overline{\nabla}_{Q\otimes\mathbb{L}}(v) = 0$

(We can always find such v that $z(v)\neq 0$)

2) $\mathbb{L} = \mathbb{L}^{1,0}\oplus\mathbb{L}^{0,1}$, $Q = Q^{0,0}$

Up to replacing $\mathbb{L}$ with $\overline{\mathbb{L}}$ we may assume

$$\dim \mathbb{L}^{1,0}_m \geqslant \dim \mathbb{L}^{0,1}_m$$

N.B.: It is not necessary that $\dim \mathbb{L}^{1,0} = \dim \mathbb{L}^{0,1}$, since $\mathbb{L}\subset\overline{\mathbb{L}}$ might be not real

Then $rk\,\overline{\nabla}_{Q\otimes\mathbb{L}}(v) = rk\,\overline{\nabla}_{\mathbb{L}}(v) \leq \dim(\mathbb{L}_m/F^1\mathbb{L}_m) =$

$= \dim \mathbb{L}^{0,1}_m \leq \frac{rk\,\mathbb{L}}{2}$

- It is important that since Q is constant, $rk\,\overline{\nabla}_{Q\otimes\mathbb{L}} = rk\,\overline{\nabla}_{\mathbb{L}}$!

6. Proof of the Theorem

Proof of Thm 1.7.1:

From Lemma we know that there exists $v \in F^1\mathcal{H}$, such that $\mathrm{rk}\,\overline{\nabla}_{\mathcal{H},m}(v) \leq \frac{\mathrm{rk}\,\mathbb{L}}{2}$.

From the identification $\overline{\nabla} = B_E(v,-)$ we get $\mathrm{rk}\,B_E(v,-) \leq \frac{\mathrm{rk}\,\mathbb{L}}{2}$, and by Prop. 4.2.3.

$$\mathrm{rk}\,E > g - \frac{\mathrm{rk}\,\mathbb{L}}{2}$$

$$\Updownarrow$$

$$2r - 2g > -\mathrm{rk}\,\mathbb{L}$$

$$\Updownarrow$$

$$\mathrm{rk}\,\mathbb{L} > 2g - 2r. \qquad \boxed{\xi}$$

We also would like to deduce the following vanishing result:

Theorem (5.2.1): Assume now that $\mathbb{V}$ is unitary only in one point, i.e. $\exists m \in \mathcal{M}: \mathbb{V}|_{C_m^0}$ is unitary. and $\mathrm{rk}\,\mathbb{V} < g$.

Then $H^0(\mathcal{M}, R^1\pi_*^0\mathbb{V}) = 0$.

Preliminaries from representation theory:

(2.1.3.)

Prop: Let $\mathcal{C} \xrightarrow{\pi} \mathcal{M}$ be a versal family of (g,n)-curves, $\rho : \pi_1(\mathcal{C}^\circ) \to GL_r(\mathbb{C})$ is a representation. Then $\rho|_{C^\circ_m}$ is MCG-finite $\forall m \in \mathcal{M}$.

s_i

$$1 \to PMod_{g,n} \to MCG_{g,n} \to \mathfrak{S}_n \to 1$$
$$\| \qquad\qquad\qquad\qquad$$
$$\pi_1(\mathcal{M}_{g,n}) \qquad\qquad\qquad$$

(action on punctures)

and

$$1 \to \pi_1(C^\circ_m) \to \pi_1(\mathcal{C}^\circ) \to \pi_1(\mathcal{M}) \to 1$$

(homotopy exact sequence)

$$1 \to \pi_1(C^\circ_m) \to \pi_1(\mathcal{M}_{g,n+1}) = PMod_{g,n+1} \to \pi_1(\mathcal{M}_{g,n}) = PMod_{g,n} \to 1$$

(Birman exact sequence)

Fact: If $\mathcal{M} \to \mathcal{M}_{g,n}$ is dominant étale, $\mathrm{im}(\pi_1(\mathcal{M}) \to \pi_1(\mathcal{M}_{g,n}))$ has finite index in $\pi_1(\mathcal{M}_{g,n}) = PMod_{g,n}$.

Now, we have an action $\pi_1(\mathcal{M}_{g,n}) \to Out(\pi_1(C^\circ_m))$.
Since the action $\pi_1(\mathcal{M}) \to \pi_1(C^\circ_m)$ preserves ρ, its image in $\pi_1(\mathcal{M}_{g,n})$ preserves ρ, but it is a finite index

subgroup ∎

(2.5.1)
Lemma: Let G be a group, $H \trianglelefteq G$ - normal
$\rho : G \to GL_r(\mathbb{C})$ a representation,
- $\det \rho$ is finite
- $\rho|_H$ is irreducible.

If $\rho|_H$ is unitary, then ρ is unitary.

Proof: Let h be a Hermitian form on $\rho|_H$.

Well if V is underlying vector space,
$h : V \xrightarrow{\sim} \overline{V}^\vee$.

By Schur lemma such h is unique up to scaling (V is an irreducible H-module),
$\forall g \in G$ $h^g : (v,w) \longmapsto h(\rho(g)v, \rho(g)w)$ is another Hermitian form, thus $h^g = \chi(g) \cdot h$, where $\chi : G \to \mathbb{C}^\times$ is a character. But $\det \rho$ is finite, hence $\chi \in \mathrm{Hom}(G, U(1))$ and ρ is unitary ∎

Decomposition of a local system unitary in one point.

Lemma (2.5.2.): Let $\mathbb{V} \to C^\circ$ be a local system,
assume $\mathbb{V}|_{C^\circ_m}$ is unitary.

There is a dominant étale base change

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "\pi'"] & \mathcal{C} \arrow[d, "\pi"] \\ \mathcal{M}' \arrow[r] \arrow[u, bend left, "s_i'"] & \mathcal{M} \arrow[u, bend left, "s_i"] \end{tikzcd}$$

with $\pi'^{0} : \mathcal{C}'^{0} \to \mathcal{M}'$ the associated family of punctured curves, s.t.

$$\mathbb{V}|_{\mathcal{C}'^{0}} = \bigoplus_{i=1}^{s} \mathbb{U}_i \otimes (\pi'^{0})^* \mathbb{W}_i ,$$

where $\mathbb{W}_i \in Loc(\mathcal{M}')$, $\mathbb{U}_i \in Loc_{U_r}(\mathcal{C}'^{0})$.

Moreover, $\bullet\ \forall m \in \mathcal{M}'$ $\mathbb{U}_i|_{C_i'^{0}}$ is irreducible,

$\bullet\ \mathbb{U}_i \not\simeq \mathbb{U}_j$ for $i \neq j$, and $\mathbb{W}_i \simeq \pi_*'^{0} Hom(\mathbb{U}_i, \mathbb{V}|_{\mathcal{C}'^{0}})$.

Proof: Since $\mathbb{V}|_{C_m^0}$ is unitary, it is semi-simple:

$$\mathbb{V}|_{C_m^0} \simeq \bigoplus_{i=1}^{s} \mathbb{S}_i^{\oplus n_i}$$ – decomposition to irr. summands

Let ρ be the monodromy of $\mathbb{V}$ and $\rho = \bigoplus_{i=1}^{s} \rho_i^{\oplus n_i}$.
By Proposition 2.1.3. ρ is MCG-finite. But then each ρ_i is MCG-finite. Indeed $\forall \gamma \in MCG\ \exists t$: $\gamma^t \cdot [\rho] \simeq [\rho]$, hence $\gamma^t[\rho_i]$ is a subquotient of $[\rho_j]$, $\gamma^t[\rho_i] \simeq [\rho_j]$. But the set of indices is finite.

Claim: $\exists\ \mathcal{M}' \to \mathcal{M}$ and $\rho_i' : \pi_1(\mathcal{C}^0) \to GL_r(\mathbb{C})$ s.t.

$\forall i \ \rho'_i|_{\mathcal{C}^0_{m'}}$ is identified with ρ_i

We would like to extend each ρ_i to a global representation $\rho'_i : \pi_1(\mathcal{C}^0) \longrightarrow GL_r(\mathbb{C})$.

Step 1: Since ρ_i is MCG-finite, after finite étale base change $MCG \cdot [\rho_i] = \{[\rho_i]\}$ (it is fixed)

But $Aut([\rho_i]) = \mathbb{C}^\times$, so it only extends to a projective rep. $\tilde{\rho}_i : \pi_1(\widetilde{\mathcal{C}^0}) \longrightarrow PGL_r(\mathbb{C})$

Step 2: $\det \rho_i$ is finite $\Rightarrow$ extends to ρ'_i after another covering.

Let $\mathbb{U}_i$ be the corr. loc. sys. By Lemma 2.5.1 they're unitary. There is a map $\bigoplus_{i=1}^{s} (\pi'^0)^* \mathbb{W}_i \otimes \mathbb{U}_i \to \mathbb{V}$, $\mathbb{W}_i$ as above.

It is fiberwise an isomorphism $\Rightarrow$ an iso $\boxtimes$

Proof of Theorem 5.2.1:

If $M' \longrightarrow M$ is dominant,

$$H^0(M, R^1\pi^0_* \mathbb{V}) \hookrightarrow H(M', R^1\pi^0_* \mathbb{V}|_{M'}) \text{ is injective.}$$

Hence we can assume:

$$\mathbb{V} = \bigoplus_{i=1}^{s} \mathbb{U}_i \otimes (\pi^0)^* \mathbb{W}_i \ , \quad \mathbb{U}_i \text{ are unitary}, \ \mathbb{W}_i \text{ are in } Loc(M)$$

Enough to show:

$$H^0(\mathcal{M}, R^1\pi^\circ_* (\mathbb{U}\otimes(\pi^\circ)^*\mathbb{W})) = H^0(\mathcal{M}, R^1\pi^\circ_*\mathbb{U}\otimes\mathbb{W}) = 0$$

if $\mathrm{rk}(\mathbb{U}\otimes\mathbb{W}) < g$.

Let $0 \neq \alpha \in H^0(\mathcal{M}, R^1\pi^\circ_*\mathbb{U}\otimes\mathbb{W})$. It can be regarded as:

$$\alpha : \mathbb{W}^\vee \longrightarrow R^1\pi^\circ_*\mathbb{U}.$$

If $\mathbb{L} = \mathrm{Im}\,\alpha$, by Theorem 1.7.1: $\mathrm{rk}\,\mathbb{L} \geq 2g - 2\,\mathrm{rk}\,\mathbb{U}$,

hence $\mathrm{rk}\,\mathbb{W} \geq \mathrm{rk}\,\mathbb{L} > 2g - 2\,\mathrm{rk}\,\mathbb{U}$.

Thus: $\mathrm{rk}\,\mathbb{W} + 2\,\mathrm{rk}\,\mathbb{U} \geq 2g$. We are interested in

$\mathrm{rk}(\mathbb{U}\otimes(\pi^\circ)^*\mathbb{W}) = \mathrm{rk}\,\mathbb{U}\cdot\mathrm{rk}\,\mathbb{W}$

Exercise: If $w, u \in \mathbb{N}^\times$, $w + 2u \geq 2g$, then $w\cdot u \geq g$

$\left(w\cdot u \geq u,\quad u \geq \left\lceil \frac{2g-1}{2} \right\rceil \geq g\right)$

$wu \geq u(2g-2u) = 2u(g-1) \geq 2g-2 \geq g$, since $g \geq 2$.

Appendix: Let $\mathbb{V}$ be a real variation of Hodge structures, $\mathbb{L} \subset \mathbb{V}_{\mathbb{C}}$ a local system. Assume that $\mathbb{L}$ is irreducible. How far is $\mathbb{L}$ from being a VHS?

1) Assume $\mathbb{V}$ is pure. By Deligne, it splits

$$\mathbb{V} = \bigoplus_i \mathbb{S}_i \otimes V_i,$$ where $\mathbb{S}_i$ are simple VHS, V_i are Hodge structures.

$\exists i: \mathbb{L} \simeq \mathbb{S}_i$. This defines a VHS-str. on $\mathbb{L}$. However, the natural map $\mathbb{L} \hookrightarrow \mathbb{V}$ is not a morphism of VHS. Consider, e.g. the case when

$\mathbb{L} = \mathbb{S}_i \otimes v$, $v \in V_i$ - is not a Hodge vector (not of type $(0,0)$).

If one put $Q := \mathrm{Hom}_{\text{Loc. Sys.}}(\mathbb{L}, \mathbb{V}) = H^0(X, \mathbb{L}^\vee \otimes \mathbb{V})$, this carries a Hodge structure (by fixed part theorem) and the natural morphism

$$Q \otimes \mathbb{L} \to \mathbb{W}$$

is a morphism of VHS. (in fact $Q = V_i$ as VHS)

2) Assume $\mathbb{V}$ is mixed. It is true that:

1) $\exists j$ s.t. the projection $\mathbb{L} \to gr_W^j \mathbb{V}$ is non-zero.

2) There exists a decomposition

$$gr_W^j \mathbb{V} = \bigoplus \mathbb{S}_i \otimes V_i$$

and $\mathbb{L} \simeq \mathbb{S}_i$ for some i.

This defines a VHS-str. on $\mathbb{L}$

3) If $\mathbb{W}$ is admissible, then

$Q := \mathrm{Hom}_{\text{Loc.Sys.}}(\mathbb{L}, \mathbb{W}) = H^0(X, \mathbb{L}^\vee \otimes \mathbb{W})$ carries a canonical Hodge structure

BUT the morphism $Q \otimes \mathbb{L} \to \mathbb{W}$ is not a morphism of VHS.

Here is the reason: let V be a mixed Hodge structure s.t. the weight filtration has two steps, but do not split

$$\left(\begin{array}{l} V \neq gr^0_W V \oplus gr^1_W V \text{ as Hodge structures} \\ \text{but} \simeq \text{ as vector spaces} \end{array} \right)$$

Let X be a compact complex manifold and $\mathbb{W} := V \otimes \mathbb{C}_X$ a constant VHS.

Then $\mathbb{V} \simeq gr^0_W \mathbb{V} \oplus gr^1_W \mathbb{V}$ in category local systems but if $\mathbb{L} = gr^1_W \mathbb{V}$, the natural morphism

$Q \otimes \mathbb{L} \to \mathbb{V}$ does not have to be a morphism of MVHS.

However, there is a situation when it is still true. For instances if both W° and F° have two steps (this is the case in Landesman-Litt's paper).

In this case $\mathcal{H} := \mathbb{V} \otimes \mathcal{O}_X$ splits as

$$\mathcal{H} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \oplus \mathcal{H}^{1,1}$$

(in cat. of vector bundles)

and $\mathcal{H}^{1,0} = W_1 \cap F^1$

$\mathcal{H}^{0,1} = W_1 \cap \bar{F}^1$

$\mathcal{H}^{1,1} = F^1 \cap \bar{F}^1$.

Thus, if $\mathbb{L} \subset \mathbb{V}$ a local sub-system:

1) if $\mathbb{L} \subset W_1\mathbb{V}$, then $\mathbb{L}$ is a local subsystem inside pure sub-VHS and the considerations as above apply.

2) otherwise, $\mathbb{L} \simeq gr^1_W \mathbb{V}$, that can be lifted to $\mathbb{V}$ as $gr^1_W\mathbb{V} \xrightarrow{\sim} F^1\mathbb{V} \cap \bar{F}^1\mathbb{V}$. This is a lifting in category of Hodge structures, so $\mathbb{L}$ is isomorphic to a sub-VHS in $\mathbb{V}$.